Operations Research Letters 44 (2016) 779–783

Contents lists available at ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

Mathematical modeling and analysis of insolvency contagion in an interbank network

Xuemin Ren *, Lishang Jiang

Department of Mathematics, TongJi University, Shanghai, 200092, China

ARTICLE INFO

Article history:
Received 28 June 2014
Received in revised form
30 September 2016
Accepted 30 September 2016
Available online 11 October 2016

Keywords:
Systemic risk
Clearing payment vector
Contagion
Interbank network
Double obstacle problem

ABSTRACT

In 2001, Eisenberg and Noe introduced a concept called "clearing payment vector" to handle the central clearing procedure of an interbank network. In this paper, we explore several conditions on the uniqueness of the clearing payment vector. It is shown that the clearing payment vector is, in general, not unique, but rather the net value of a bank is. Then, we show that the net value of the bank continuously depends on operating cash flows and so does the clearing payment vector under the condition that ensures the uniqueness of the clearing payment vector.

1. Introduction

Since the early 2000s, numerous network models of interbank have been developed and used in analysis of insolvency contagion and systemic risk. The term "contagion" represents a domino effect of the insolvency of banks. In [3], Elsinger, Lehar, and Summer made a survey of these models and discussed their main accomplishments. The reader is also referred to [6] for an overview of different contagion channels.

In [2], Eisenberg–Noe established a model that was not only used to derive an existence condition for the central clearing mechanism of interbank networks, but also served the purpose of investigating propagation of defaults through a network of interbank liabilities. Following their terminology, the "liabilities" are described by an $n \times n$ liability matrix L, and the vector of operating cash flow e belongs to

$$\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) : x_j \geq 0 \text{ for all } 1 \leq j \leq n\},$$

where n is the number of nodes (banks) in a network. Eisenberg–Noe [2] proved that clearing payment vectors exist for every $e \in \mathbb{R}^n_+$ and, under some mild conditions on network structures, there is only one vector of such. Elsinger–Lehar–Summer [4] then improved Eisenberg–Noe's original model to handle the case in which the operating cash flow vector e is in $\mathbb{R}^n$. There, however, a bilateral obstacle problem arises for clearing payment vectors when e lies in $\mathbb{R}^n$. In [1], Acemoglu et al. used Brouwer's Fixed Point theorem to prove the existence of the clearing payment vector, and they were further able to obtain a sufficient condition for its uniqueness.

Our main purpose in this paper is to generalize the well-posed model established by Elsinger–Lehar–Summer [4]. We prove that the clearing payment vector, which is the solution of the bilateral obstacle problem, is in general not unique, but rather the net value of a bank is, with stability, with respect to the change of the operating cash flow. Further, we extend the uniqueness condition for the clearing payment under a regularity condition in reference to the network structure introduced by Eisenberg–Noe [2]. Computations of systemic risks solely rely on the hypothesis of the distribution of the operating cash flows and Monte Carlo simulations. With this in mind, it is extremely important to note that the computations are not reliable without imposing special conditions, such as clearing payment vectors or the fact that net values continuously depend on the operating cash flows.

This paper is organized as follows: in Section 2, the general framework of financial systems in an interbank network is described. Next, Section 3 focuses on the existence and uniqueness of net values and clearing payment vectors. Then in Section 4, we discuss some continuity properties for net values and clearing payment vectors.

* Corresponding author.
E-mail address: ren-xuemin@163.com (X. Ren).

http://dx.doi.org/10.1016/j.orl.2016.09.017
0167-6377/

2. General framework

Consider a financial network populated by n financial institutions, or financial nodes, among which one node may have nominal liabilities to other nodes in the network represented by the nominal liability matrix $L = \left(L_{ij}\right)_{n\times n}$, where L_{ij} is the nominal liability of node i to node j. Obviously, $L_{ii} = 0$ for any $1 \le i \le n$. It is also clear that $L_{ij} \ge 0$ and usually $L_{ij} \ne L_{ji}$ for any i, j with $i \ne j$ and $1 \le i, j \le n$. Hence L is a non-negative and non-symmetric matrix with all diagonal entries being zero.

Let e_i denote an exogenous operating cash flow of node i (here $e_i \ge 0$ means that a cash flow is received; $e_i < 0$ means that a cash flow is paid). The tuple $e = (e_1, \dots, e_n)$ is called a vector for the exogenous operating cash flows. A financial system is defined as a pair (L, e) that consists of both inside and outside cash flows. Let p_i be the total real payment of node i to all other nodes in the system. We call $p = (p_1, \dots, p_n)$ a clearing payment vector. Let $\bar{p}_i$ be the total liability of node i to all other nodes; that is, $\bar{p}_i = \sum_{j=1}^{n} L_{ij}$. Write $\bar{p} = (\bar{p}_1, \dots, \bar{p}_n)$ and call $\bar{p}$ a nominal liability vector. Let

$$\Pi_{ij} = \begin{cases} \dfrac{L_{ij}}{\bar{p}_i}, & \text{if } \bar{p}_i > 0, \\ 0 & \text{if } \bar{p}_i = 0. \end{cases}$$

The matrix $\Pi = (\Pi_{ij})_{n\times n}$ is called a relative liability matrix. It is easy to verify that for any $1 \le i \le n$, we have $\sum_{j=1}^{n} \Pi_{ij} = 1$. Thus Π is a transition matrix and its largest eigenvalue must be one. Throughout the paper, by a clearing payment vector for the financial system $(\Pi, \bar{p}, e)$ we mean a vector $p^* \in [0, \bar{p}]$ that satisfies the following three conditions:

(1) (Limited liability) the total payment does not exceed its asset,
(2) (Absolute priority) for every node i, either its obligation is paid fully, i.e., $p_i = \bar{p}_i$, or all assets are paid to creditors, i.e., $0 \le p_i < \bar{p}_i$, and
(3) (Same priority) all obligations have the same priority; that is, all claimant nodes are paid by the default node in proportion to the size of their nominal claim on firms' assets, so that the real payment of node i to node j is $p_i \Pi_{ij}$.

For a given clearing payment vector $p = (p_1, \dots, p_n)$, node i receives a payment from network $\sum_{j=1}^{n} \Pi_{ij}^T p_j$ so that the total income of node i is $\sum_{j=1}^{n} \Pi_{ij}^T p_j + e_i$ and the total outcome of node i is p_i, where we denote $\Pi_{ij}^T = \Pi_{ji}$. For each i, the clearing payment vector p, $0 \le p \le \bar{p}$, satisfies one of the following conditions:

$$\begin{cases} p_i = \bar{p}_i \\ \left(\Pi^T p + e - p\right)_i \ge 0, \end{cases} \tag{2.1}$$

or

$$\begin{cases} 0 < p_i < \bar{p}_i \\ \left(\Pi^T p + e - p\right)_i = 0, \end{cases} \tag{2.2}$$

or

$$\begin{cases} p_i = 0 \\ \left(\Pi^T p + e - p\right)_i \le 0. \end{cases} \tag{2.3}$$

Expression (2.1) says that the total income of node i can cover all of the obligations and the equity value is non-negative. Expression (2.2) says that the total income of node i can just cover a part of the obligations and the equity value vanishes. Expression (2.3) tells us that the total income of node i is non-positive and the equity value vanishes, and this occurs only when $e_i \le 0$. Obviously, (2.1)–(2.3) combine to yield a more concise form

$$\min\left\{\bar{p} - p, \ \max\left\{\left(\Pi^T - I\right)p + e, \ -p\right\}\right\} = 0, \tag{2.4}$$

or equivalently,

$$p = \min\left\{\bar{p}, \ \max\left\{\left(\Pi^T p + e\right), \ 0\right\}\right\}. \tag{2.5}$$

Remark 2.1. The model in [2] only contains expressions (2.1) and (2.2) whose condensed expression is

$$p = \min\left\{\bar{p}, \left(\Pi^T p + e\right)\right\},$$

which is a single obstacle problem. Expression (2.4) suggests that the main difficulty is the degeneration of the matrix $\Pi^T - I$.

3. Existence and uniqueness of clearing payment vectors

In this section, we discuss existence and uniqueness conditions for clearing payment vectors. Define a map $\Phi(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ by the formula

$$\Phi(p; \Pi, \bar{p}, e) = \min\left\{\bar{p}, \ \max\left\{\left(\Pi^T p + e\right), \ 0\right\}\right\}.$$

It is well-known from [4] that any clearing payment vector is a fixed point of the map $\Phi(p)$.

Definition. A financial network is said to be regular if for any nodes i and j, $1 \le i, j \le n$, there is a path $(i_1 i_2 \cdots i_m)$ that joins from node i to node j such that $i_1 = i$, $i_m = j$, and $\prod_{i_l i_{l+1}} > 0$ for $1 \le l \le m-1$.

By the term "regular financial network" we mean that for any two financial institutions in the network, there exists a direct or indirect obligation between them. This is equivalent to, according to graph theory, a unilaterally strong connected graph with irreducible representation matrix.

Theorem 3.1 (*Acemoglu–Ozdaglar–Tahbaz-Salehi [1]*).

(1) *In a given financial system $(\Pi, \bar{p}, e)$, there exist a greatest clearing payment vector p^+ and a least clearing payment vector p^-.*
(2) *For a regular financial network with the property that the sum of all operating cash flows $\sum_{i=1}^{n} e_i \ne 0$, the clearing payment vector p^*, $0 \le p^* \le \bar{p}$, is unique.*

We remark that the condition $\sum_{j=1}^{n} e_j \ne 0$ is just a sufficient condition for the uniqueness of the clearing payment vector under the regularity condition.

In the case where $\sum_{j=1}^{n} e_j = 0$, the uniqueness of the clearing payment vector is not guaranteed at all. For example, we take $\Pi^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and assume that the nominal obligations $\bar{p}_1 = \bar{p}_2 = 2$.

Case 1. $e_1 = -1$ and $e_2 = 1$. In this case, we have

$$\Pi^T p^* + e - p^* = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} p_1^* \\ p_2^* \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 0,$$

which simplifies to

$$\begin{cases} -p_1^* + p_2^* = 1 \\ p_1^* - p_2^* = -1. \end{cases}$$

It is easy to see that $p_1^* = t$ and $p_2^* = 1 + t$, $0 \le t \le 1$, are the solutions of the system. So the solution is not unique.

Case 2. $e_1 = -2$ and $e_2 = 2$. In this case, we have

$$\begin{cases} -p_1^* + p_2^* = 2 \\ p_1^* - p_2^* = -2. \end{cases}$$

The solutions for the system are $p_1^* = t$ and $p_2^* = 1 + t$ for $0 \le t \le 1$, which satisfy the conditions $0 \le p_1^* \le 2$ and $0 \le p_2^* \le 2$. So we must have $t = 0$. In this case, the solution is unique.