



Multi-period portfolio optimization: Translation of autocorrelation risk to excess variance



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ABSTRACT

Growth-optimal portfolios are guaranteed to accumulate higher wealth than any other investment strategy in the long run. However, they tend to be risky in the short term. For serially uncorrelated markets, similar portfolios with more robust guarantees have been recently proposed. This paper extends these robust portfolios by accommodating non-zero autocorrelations that may reflect investors' beliefs about market movements. Moreover, we prove that the risk incurred by such autocorrelations can be absorbed by modifying the covariance matrix of asset returns.

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1. Introduction

In this paper, we consider a dynamic portfolio optimization problem where investors face a challenge of how to allocate their wealth over a set of available assets to maximize their terminal wealth. By optimizing expected log-utility over a single investment period, the obtained portfolio, referred to as the growth-optimal portfolio, is shown to be optimal with respect to several interesting objectives in a classical stochastic setting. For example, [5,2] independently demonstrate that the growth-optimal portfolio will eventually accumulate more wealth than any other causal investment strategy with probability 1 in the long run. Moreover, it also minimizes the expected time required to reach a specified wealth target when the target is asymptotically large; see, e.g., [2,1]. For the readers interested in the history and the properties of the growth-optimal portfolio, we refer to [3,9,10]. Nonetheless, despite its theoretical appeals, there are many reasons why the practical relevance of the growth-optimal portfolio remains limited.

First, empirically the growth-optimal portfolio is shown to be highly volatile in the short run. Moreover, the calculation of the growth-optimal portfolio requires full and precise knowledge of the asset return distribution. In practice, this distribution is not available and has to be estimated from sparse empirical data.

Therefore, the growth-optimal portfolio is prone to statistical estimation errors. [12] extends the growth-optimal portfolio to more practical settings by proposing a fixed-mix investment strategy that offers a similar performance guarantee as the classical growth-optimal portfolio but for a finite investment horizon. Moreover, the proposed performance guarantee is not distribution-specific but remains valid for any asset return distribution within the prescribed ambiguity set of serially uncorrelated distributions.

Our contribution in this paper is to extend the results in [12] to the case with non-zero autocorrelations (also known as, serial correlations). These autocorrelations can be used to incorporate beliefs of the investors about market movements as well as seasonality in asset returns; see e.g. [13,7]. Moreover, we prove that these autocorrelations can be absorbed in the covariance matrix underlying the asset return distribution. Finally, we remark that all of the discussed *dynamic* investment strategies, namely the classical growth-optimal portfolio, the robust growth-optimal portfolio (see [12]), and the extended robust growth-optimal portfolio (proposed in this paper), share similar computational advantage, in the sense that, all of them can be obtained with relative ease by solving *static* optimization problems.

The rest of the paper is structured as follows. In Section 2 we explain how we model the distributional ambiguity in financial markets, and in Section 3 we define the risk measure, namely worst-case growth rate, to assess the performance of each individual portfolio. An analytical formula for the worst-case growth rate is then derived in Section 4. A couple of numerical experiments are also given in this section. Finally, we

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provide an approximate worst-case growth rate for a more general probabilistic setting in Section 5.

Notation. We denote the space of symmetric matrices in $\mathbb{R}^{n \times n}$ by $\mathbb{S}^n$. For any symmetric matrices $\mathbf{X}$ and $\mathbf{Y}$ with the same dimension, we denote their trace scalar product by $\langle \mathbf{X}, \mathbf{Y} \rangle$. Moreover, for a positive semidefinite $\mathbf{X} \in \mathbb{S}^n$, we define $\mathbf{X}^{1/2}$ as its principle square root. We also define $\mathbf{1}$ as a column vector of ones and $\mathbf{I}$ as an identity matrix. Their dimensions should be clear from the surrounding context. Random variables are represented by symbols with tildes. We denote by $\mathcal{P}_0^n$ the set of all probability distributions $\mathbb{P}$ on $\mathbb{R}^n$, and we represent by $\mathbb{E}_{\mathbb{P}}(\cdot)$ and $\text{COV}_{\mathbb{P}}(\cdot, \cdot)$ the expectation and the covariance of the input random parameter(s) with respect to the probability distribution $\mathbb{P}$. Throughout the paper, we assume that the investment horizon is given by $\mathcal{T} = \{1, 2, \dots, T\}$ and the asset universe is given by $\mathcal{N} = \{1, 2, \dots, N\}$. Moreover, for a complex number c , we denote its real part by $\text{Re}(c)$. Finally, we define $(t)_T$ as the residue of t modulo T . Note that for any $t \in \mathbb{Z}$, $(t)_T$ takes a value from $\{0, \dots, T-1\}$.

2. Distributional setting: stationary means, variances, and autocorrelations

In this section, we describe the setting of the probability distributions $\mathbb{P}$ of the asset returns $[\tilde{\mathbf{r}}_t]_{t=1}^T$. We define the ambiguity set

$$\mathcal{P} = \left\{ \mathbb{P} \in \mathcal{P}_0^{NT} : \begin{array}{l} \mathbb{E}_{\mathbb{P}}(\tilde{\mathbf{r}}_{t,i}) = \mu_i \quad \forall i \in \mathcal{N} \quad \forall t \in \mathcal{T} \\ \text{COV}_{\mathbb{P}}(\tilde{\mathbf{r}}_{s,i}, \tilde{\mathbf{r}}_{t,j}) = \rho_{(t-s)_T} \sigma_{i,j} \\ \forall i, j \in \mathcal{N} \quad \forall s, t \in \mathcal{T} \end{array} \right\},$$

where $\boldsymbol{\mu} = [\mu_i]_{i \in \mathcal{N}} \in \mathbb{R}^N$ stands for the vector of expected asset returns and $\boldsymbol{\Sigma} = [\sigma_{i,j}]_{i,j \in \mathcal{N}} \in \mathbb{S}^N$ stands for the covariance matrix of asset returns. Both $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are assumed to be stationary, i.e., they remain unchanged over time. This choice of ambiguity set $\mathcal{P}$ is nicely motivated in [11]. In contrast to [12] which assumes uncorrelatedness between asset returns at different trading periods s and t , we allow them to be correlated with correlation $\rho_{(t-s)_T}$.

Next, we consider a fixed-mix strategy (see [12]) generated by a portfolio $\mathbf{w} \in \mathcal{W} \subset \mathbb{R}_+^N$, where $\mathcal{W}$ is a convex polyhedral of allowable portfolios. That is, we assume that portfolio weights revert to $\mathbf{w}$ at every rebalancing date $t \in \mathcal{T}$. In order to avoid clutter, we denote the portfolio return during the trading period t by $\tilde{\eta}_t = \mathbf{w}^\top \tilde{\mathbf{r}}_t$. It is clear that, under the distribution $\mathbb{P}$ of asset returns, we have that

$$\mathbb{E}_{\mathbb{P}}(\tilde{\eta}_t) = \mathbf{w}^\top \boldsymbol{\mu} \quad \text{and} \quad \text{COV}_{\mathbb{P}}(\tilde{\eta}_s, \tilde{\eta}_t) = \rho_{(t-s)_T} \mathbf{w}^\top \boldsymbol{\Sigma} \mathbf{w} \quad \forall s, t \in \mathcal{T}.$$

Equivalently put, the mapping $\tilde{\eta}_t = \mathbf{w}^\top \tilde{\mathbf{r}}_t$ projects $\mathcal{P}$ to an ambiguity set $\mathcal{P}(\mathbf{w})$ defined as

$$\mathcal{P}(\mathbf{w}) = \left\{ \mathbb{P} \in \mathcal{P}_0^T : \begin{array}{l} \mathbb{E}_{\mathbb{P}}(\tilde{\eta}_t) = \mathbf{w}^\top \boldsymbol{\mu} \quad \forall t \in \mathcal{T} \\ \text{COV}_{\mathbb{P}}(\tilde{\eta}_s, \tilde{\eta}_t) = \rho_{(t-s)_T} \mathbf{w}^\top \boldsymbol{\Sigma} \mathbf{w} \\ \forall s, t \in \mathcal{T} \end{array} \right\}.$$

The projection property of the ambiguity set (see, for example, [15, Theorem 1] and [12, Proposition 2]) further asserts the existence of the inverse mapping from $\mathcal{P}(\mathbf{w})$ to $\mathcal{P}$. Hence, from the next section onward, we will study the performance of an arbitrary portfolio $\mathbf{w}$ using the projected ambiguity set $\mathcal{P}(\mathbf{w})$ instead of the original ambiguity set $\mathcal{P}$, since maneuvering $\mathcal{P}(\mathbf{w})$ often leads to an optimization problem with a smaller dimension.

3. Portfolio performance measure: worst-case growth rate

In a friction-less market, a fixed portfolio $\mathbf{w}$ repeatedly invested over the investment horizon $\mathcal{T}$ leads to a total return of

$$\prod_{t \in \mathcal{T}} (1 + \mathbf{w}^\top \tilde{\mathbf{r}}_t) = \exp \left(\sum_{t \in \mathcal{T}} \log(1 + \mathbf{w}^\top \tilde{\mathbf{r}}_t) \right),$$

which is a random amount. An intuitive performance measure for this portfolio would thus be an expectation of its logarithmic terminal wealth $\mathbb{E}(\sum_{t \in \mathcal{T}} \log(1 + \mathbf{w}^\top \tilde{\mathbf{r}}_t))$. A portfolio that maximizes such utility function is referred to as a growth-optimal portfolio. This portfolio exhibits many intriguing asymptotic properties, and some of them are discussed in Section 1. Moreover, if the asset return distribution is serially independent and identically distributed, then the growth-optimal portfolio can be obtained by solving a static optimization problem $\max_{\mathbf{w} \in \mathcal{W}} \mathbb{E}(\log(1 + \mathbf{w}^\top \tilde{\mathbf{r}}_1))$. However for a finite T (especially when T is small), the expectation criterion becomes risky (as the central limit theorem fails) and accordingly [12] proposes to use a quantile criterion instead of the expectation criterion. Precisely speaking, [12] approximates $\log(1 + \mathbf{w}^\top \tilde{\mathbf{r}}_t)$ by a second-order Taylor expansion around $\mathbf{w}^\top \tilde{\mathbf{r}}_t = 0$ and employs recent advances in distributionally robust optimization (see [16]) to determine the worst-case growth rate by solving the following optimization problem

$$\begin{aligned} g_\epsilon(\mathbf{w}) &= \max_{\gamma} \left\{ \gamma : \mathbb{P} \left(\frac{1}{T} \sum_{t \in \mathcal{T}} \left(\mathbf{w}^\top \tilde{\mathbf{r}}_t - \frac{1}{2} (\mathbf{w}^\top \tilde{\mathbf{r}}_t)^2 \right) \geq \gamma \right) \geq 1 - \epsilon \right. \\ &\quad \left. \forall \mathbb{P} \in \mathcal{P} \right\} \\ &= \max_{\gamma} \left\{ \gamma : \mathbb{P} \left(\frac{1}{T} \sum_{t \in \mathcal{T}} \left(\tilde{\eta}_t - \frac{1}{2} \tilde{\eta}_t^2 \right) \geq \gamma \right) \geq 1 - \epsilon \right. \\ &\quad \left. \forall \mathbb{P} \in \mathcal{P}(\mathbf{w}) \right\} \end{aligned}$$

where the ambiguity set in [12] is the restriction of ours where $\rho_t = 0$ for every $t = 1, \dots, T-1$. The uncorrelatedness assumption allows [12] to solve this distributionally robust program efficiently because of the inherent temporal symmetry. Our work relaxes this assumption in order to accommodate investors' beliefs and market seasonality. In particular, we show that, despite the fact that the temporal symmetry breaks down, we can still derive an analytical expression of $g_\epsilon(\mathbf{w})$ by using knowledge from linear algebra of circulant matrices; see e.g. [4]. We highlight that even though the relaxation does not change the problem greatly, it still requires us to develop new mathematical techniques to accommodate these changes.

4. Derivation of worst-case growth rates

By using the semidefinite program reformulation for distributionally robust quadratic chance constraints provided in [16], we can rewrite $g_\epsilon(\mathbf{w})$ as

$$\begin{aligned} g_\epsilon(\mathbf{w}) &= \max_{\gamma} \quad \gamma \\ \text{s.t.} \quad & \mathbf{M} \in \mathbb{S}^{T+1}, \beta \in \mathbb{R}, \gamma \in \mathbb{R} \\ & \beta + \frac{1}{\epsilon} \langle \boldsymbol{\Omega}(\mathbf{w}), \mathbf{M} \rangle \leq 0, \quad \mathbf{M} \succeq \mathbf{0} \\ & \mathbf{M} - \begin{bmatrix} \frac{1}{2} \mathbf{I} & -\frac{1}{2} \mathbf{1} \\ -\frac{1}{2} \mathbf{1}^\top & \gamma T - \beta \end{bmatrix} \succeq \mathbf{0}, \end{aligned} \quad (1)$$

where $\boldsymbol{\Omega}(\mathbf{w}) \succ \mathbf{0}$ is the projected second-order moment matrix for a sequence of portfolio returns $[\tilde{\eta}_t]_{t \in \mathcal{T}}$ generated by any distribution residing in $\mathcal{P}(\mathbf{w})$, i.e.,

$$\boldsymbol{\Omega}(\mathbf{w}) = \left[\begin{array}{c|c} \mathbf{w}^\top \boldsymbol{\Sigma} \mathbf{w} \cdot \mathbf{P} + (\mathbf{w}^\top \boldsymbol{\mu})^2 \cdot \mathbf{1} \mathbf{1}^\top & \mathbf{w}^\top \boldsymbol{\mu} \cdot \mathbf{1} \\ \hline \mathbf{w}^\top \boldsymbol{\mu} \cdot \mathbf{1}^\top & 1 \end{array} \right]$$

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