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Bounds on the price of anarchy for a more general class of directed graphs in opinion formation games

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ABSTRACT

In opinion formation games with directed graphs, a bounded price of anarchy is only known for weighted Eulerian graphs. Thus, we bound the price of anarchy for a more general class of directed graphs with conditions intuitively meaning that each node does not influence the others more than she is influenced, where the bounds depend on such difference (in a ratio). We also show that there exists an example just slightly violating the conditions with an unbounded price of anarchy.

1. Introduction

In a society or community, individuals and their relationships form a social network. For some matter that each individual gets to express her own opinion about, individuals influence each other regarding it through such a social network. For example, the matter can be adopting a new innovation/product, and the individuals are potential users/consumers while their opinions could be tendency to adopt the innovation or the preference toward the product; for some political issue that may need public consensus, the public have different opinions or thoughts about it. In any case, an individual is often affected by her friends/neighbors in the social network when making up her mind. The opinion forming process in a social network can be naturally thought as an opinion influencing and updating dynamics. This has attracted researchers' interest a while ago in mathematical sociology, and more recently in theoretical computer science.

DeGroot [6] modeled the opinion formation process by associating each individual with a numeric-value opinion and letting the opinion be updated by a weighted average of the opinions from her friends and her own, where the weights represent how much she is influenced by her friends. This update dynamics will converge to a fixed point in which all individuals hold the same opinion, i.e., a consensus. However, we can easily observe that in the real world, the consensus is difficult to reach. Friedkin and Johnson [7] differentiated an *expressed opinion* that each individual in the networks updates over time from an *internal opinion* that each individual is born with and stays unchanged. Thus, an individual would be always influenced by her inherent belief, and the dynamics converges to a unique equilibrium, which may not be a consensus.

Bindel et al. [3] viewed the updating rule mentioned above equivalently as each player updating her expressed opinion to minimize her quadratic individual cost function, which consists of the disagreement between her expressed opinion and those of her friends, and the difference between her expressed and internal opinions. They analyzed how socially good or bad the system can be at equilibrium compared to the optimum solution in terms of the *price of anarchy* [9], i.e., the ratio of the social cost at the worst equilibrium to the optimal social cost. (Notions of equilibria will be given in Section 2.) The price of anarchy is at most 9/8 in undirected graphs and is unbounded in directed graphs (due to a star graph with a center only influencing the others but is not influenced at all by the others, or even directed bounded-degree trees with degrees high enough). Nevertheless, a bounded price of anarchy can be obtained for weighted Eulerian graphs, in particular, a tight upper bound of 2 for directed cycles and an upper bound of $d + 1$ for d-regular graphs. Another work closely related to that of Bindel et al. is by Bhawalkar et al. [2]. The individual cost functions are assumed to be "locally-smooth" in the sense of [12] and may be more general than quadratic, for example, convex. The price of anarchy for undirected graphs with quadratic

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cost functions is at most 2. They also allowed social networks to change by letting players choose k-nearest neighbors through opinion updates and bounded the price of anarchy. On the other hand, Chierichetti et al. [5] considered the games with discrete preferences, where an expressed and internal opinions are chosen from a discrete set and distances measuring "similarity" between opinions correspond to costs.

When graphs are directed, a bounded price of anarchy is only known for weighted Eulerian graphs where the total incoming weights equal to the total outgoing weights at each node [3,2], which may seem rather restricted. Thus, we are interested *to bound the price of anarchy for games with directed graphs more general than weighted Eulerian graphs* (even with just quadratic individual cost functions) in this article. Note that although the result of [2] is indeed for directed graphs and gives bounded price of anarchy, their setting is different from ours. In their model, the weights on the k neighbors are uniform, and more importantly, the k neighbors of a node is not fixed, as its action in the game includes choosing its k neighbors in addition to its opinion. Therefore, it is related to but different from what we propose to tackle here. We first bound the price of anarchy for a more general class of directed graphs with conditions intuitively meaning that each node does not influence the others more than she is influenced by herself and the others, where the bounds depend on such influence differences (in a ratio). This generalizes the previous results on directed graphs, and recovers and matches the previous bounds in some specific classes of (directed) Eulerian graphs. We then show the existence of an example that just slightly violates the conditions but with an unbounded price of anarchy. We further propose more research directions in the discussions and future work.

2. Preliminaries

We describe a social network as a weighted graph $(G, \mathbf{w})$ for directed graph $G = (V, E)$ and matrix $\mathbf{w} = [w_{ij}]_{ij}$. The node set V of size n is the selfish players, and the edge set E is the relationships between any pair of nodes. The edge weight $w_{ij} \geq 0$ is a real number and represents how much player i is influenced by player j; note that weight w_{ii} can be seen as a self-loop weight, i.e., how much player i influences (or is influenced by) herself. Each (node) player has an internal opinion s_i, which is unchanged and not affected by opinion updates. An *opinion formation game* can be expressed as an instance $(G, \mathbf{w}, \mathbf{s})$ that combines weighted graph $(G, \mathbf{w})$ and vector $\mathbf{s} = (s_i)_i$. Each player's strategy is an expressed opinion z_i, which may be different from her s_i and gets updated. Both s_i and z_i are real numbers. The individual cost function of player i is

$$C_i(\mathbf{z}) = w_{ii}(z_i - s_i)^2 + \sum_{j \in N(i)} w_{ij}(z_i - z_j)^2$$
$$= w_{ii}(z_i - s_i)^2 + \sum_{j} w_{ij}(z_i - z_j)^2,$$

where $\mathbf{z}$ is the strategy profile/vector and $N(i)$ is the set of the neighbors of i, i.e., $\{j : j \neq i, w_{ij} > 0\}$. It measures the disagreement between her expressed opinion and those of her friends, and the difference between her expressed and internal opinions. Other functions could also be used other than the square one here, for example, convex functions [2]. Each node minimizes her cost C_i by choosing her expressed opinion z_i. We analyze the game when it stabilizes, i.e., at equilibrium.

Equilibria and the price of anarchy

In a (pure) Nash equilibrium $\mathbf{z}$, each player i's strategy is z_i such that given $\mathbf{z}_{-i}$ (i.e., the opinion vector of all players except i) for any other z'_i,

$$C_i(z_i, \mathbf{z}_{-i}) \leq C_i(z'_i, \mathbf{z}_{-i}).$$

According to [3,2], in such an equilibrium, the condition

$$z_i = \frac{w_{ii}s_i + \sum_{j \neq i} w_{ij}z_j}{w_{ii} + \sum_{j \neq i} w_{ij}}$$

holds for any player i. This can be shown by taking the derivative of C_i w.r.t. z_i, setting it to 0 for each i, and solving the equality system since very player i minimizes C_i. Note that C_i is continuously differentiable.

The social cost function here is $C(\mathbf{z}) = \sum_i C_i(\mathbf{z})$, the sum of the individual costs. Nash equilibria can be far from the (centralized) social optimum in terms of a social cost [10]. To measure the (in)efficiency of equilibria, the price of anarchy [9] is defined as the ratio of the worst equilibrium's social cost to the optimal social cost.

Local smoothness

To bound the price of anarchy, the *local smoothness* framework developed by Roughgarden and Schoppmann [12] is a promising analysis technique in algorithmic game theory. It has been applied in [2] to obtain the price of anarchy bounds there and similar techniques have been used in many other games [11,4]. The local smoothness technique is slightly different from the smoothness techniques of [11,4]. The local smoothness technique is sometimes more suitable since it gives tight(er) bounds while the smoothness technique does not in some games. We briefly summarize this technique in the following. The inequality intuitively means that summing up individual costs after some unilateral local deviations to o_i's from any strategy profile $\mathbf{z}$ can still be upper bounded by a combination of the social costs $C(\mathbf{z})$ and $C(\mathbf{o})$ (where the derivative term accounts for localness), implying that $C(\mathbf{z})$ is not too far from $C(\mathbf{o})$. As in [12], we assume that the cost function C_i of each player i is continuously differentiable w.r.t. her strategy.

Definition 1. A cost-minimization game is (λ, μ)-locally smooth if for any strategy profiles $\mathbf{o}$ and $\mathbf{z}$,

$$\sum_i \left(C_i(z_i, \mathbf{z}_{-i}) + (o_i - z_i)\frac{\partial}{\partial z_i} C_i(z_i, \mathbf{z}_{-i}) \right) \leq \lambda C(\mathbf{o}) + \mu C(\mathbf{z}). \quad (1)$$

We have the following from an extension theorem of [12].

Theorem 1. *If σ is a correlated equilibrium and $\mathbf{o}$ is the social optimum in a (λ, μ)-locally smooth game, for $\lambda > 0$ and $\mu < 1$, then the correlated (thus, pure and mixed) price of anarchy, $E_{\mathbf{z}\sim\sigma}[C(\mathbf{z})]/C(\mathbf{o})$, is at most $\lambda/(1 - \mu)$.*

With this, our goal becomes finding suitable λ and μ values to satisfy Inequality (Eq. (1)), which immediately gives price-of-anarchy bounds.

3. Bounds on the price of anarchy

We first generalize the result [3] about bounded price of anarchy for weighted Eulerian graphs to a more general class of directed graphs with conditions intuitively meaning that each node does not influence the others more than she is influenced (by herself and the others). We then show that there exists an example that just slightly violates the conditions and gives an unbounded price of anarchy.

Recall from [3] that in a weighted Eulerian graph, $\sum_{j \neq i} w_{ij} = \sum_{j \neq i} w_{ji}$ for every node i, which means that the influence each node exerts on the others is exactly the same as that it receives from the others. Here we relax the condition and obtain the following.