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A matrix approach to the associated consistency with respect to the equal allocation of non-separable costs

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ABSTRACT

We adopt a *matrix* approach to show that the equal allocation of non-separable costs (EANSC) value is the only solution satisfying *Pareto optimality, translation covariance, symmetry, continuity* and *associated consistency*.

1. Introduction

This paper contributes to the growing literature on associated consistency. Associated consistency states that a solution is invariant to a certain linear transformation of games, called associated games. Hwang [3] introduced a version of associated games, called the E-associated game, and axiomatized the equal allocation of non-separable costs (EANSC) value as a unique solution for verifying Pareto optimality, translation covariance, symmetry, continuity and associated consistency. In this paper, we present a matrix approach for Hwang's axiomatization of the EANSC value.

To prove the main result, we apply certain methods of linear algebra to the theory of cooperative games. We introduce the notion of a row (resp. column) coalitional matrix in the framework of cooperative game theory. The EANSC value and the E-associated games are represented algebraically by their coalitional matrices, which are called the EANSC standard matrix and the E-associated transformation matrix, respectively. The associated consistency for the EANSC value means that the EANSC standard matrix is invariant when multiplied with the E-associated transformation matrix. The diagonalization procedure of the E-associated transformation matrix, the constant property and the inessential property of coalitional matrices are fundamental tools that prove the convergence of the sequence of repeated E-associated games as well as its limit game to be the sum of a constant game and an inessential game. Combining this, associated consistency, and continuity, demonstrates that the solution of any game is the solution of a constant game plus an inessential game. The subsequent application of Pareto optimality, translation covariance, and symmetry, can prove that the solution is uniquely determined and coincides with the EANSC value. Thus, we achieve a matrix approach for Hwang's axiomatization of the EANSC value.

Hamiache [1] was the first to introduce the notion of associated consistency and showed that the Shapley value is characterized as the unique solution satisfying associated consistency (with respect to Hamiache's associated game), continuity and the inessential game property. The associated consistency of the Shapley value is studied by applying the matrix approaches in Xu et al. [4] and Hamiache [2]. Hwang [3] modified the definition of Hamiache's associated game in order to characterize the EANSC value. He showed that the EANSC value is characterized as the unique solution satisfying associated consistency (with respect to Hwang's associated game), continuity, Pareto optimality, translation covariance, and symmetry. Recently, Xu et al. [5] defined a new version of an associated game in order to characterize the EANSC value, which is in the framework of Hamiache [1]. By the matrix approach, they showed that the EANSC value is characterized as the unique solution satisfying associated consistency (with respect to the new associated game), continuity and the inessential game property. These findings mean that we can distinguish two frameworks within the literature of associated consistency. The first one was developed by Hamiache [1] and further explored by Xu et al. [4], Hamiache [2], and Xu et al. [5], among others. In this framework, solutions are characterized by associated consistency with respect to a certain linear transformation of games, continuity, and the inessential

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game property. Within this framework, the main purpose is to construct an associated game such that the continuous solution under consideration is invariant to this transformation and such that the limit of the sequence of the repeated associated games is an inessential game. In the second framework initiated by Hwang [3], the purpose is similar except that the limit of the induced sequence of repeated associated games is the sum of an inessential game and a constant game. In this paper, by applying matrix approach, we study the associated consistency of the EANSC value, following the framework of Hwang [3].

2. Definitions and notations

The notations $S \subset T$ and $S \subseteq T$ mean that S is a proper subset of T and S is a subset of T, respectively. We use the lowercase letter s to denote the number of elements in a set S. A game with transferable utility (TU game) is a pair (N, v) where $N = \{1, 2, \ldots, n\}$ and v is a mapping such that $v : 2^N \longrightarrow \mathbb{R}$ and $v(\emptyset) = 0$. A game (N, v) is an **inessential game** if for all $S \subseteq N$, $v(S) = \sum_{i\in S} v(\{i\})$. A game (N, v) is a **constant game** if for all $S \subseteq N$ with $S \neq \emptyset$, $v(S) = c$ where $c \in \mathbb{R}$. We denote by G the set of all games in which the set of players is N. A solution on G is a function σ which associates to each game $(N, v) \in G$ an element $\sigma(N, v)$ of $\mathbb{R}^N$.

For all games (N, v) and all players j in N, the term $SC_j(N, v)$ and the term $NSC(N, v)$ mean the separable cost and the non-separable cost, respectively, where

$$SC_j(N, v) = v(N) - v(N \setminus \{j\});$$
$$NSC(N, v) = v(N) - \sum_{k\in N} SC_k(N, v).$$

The EANSC value, E, is a solution function on G, and it associates to each game (N, v) and all players j in N the value,

$$E_j(N, v) = SC_j(N, v) + \frac{1}{n}[NSC(N, v)].$$

3. Associated game and axiomatization

Hwang [3] introduced the notion of the "associated game" to investigate the EANSC value. The definition of associated game is given as follows:

Definition 1. Let (N, v) be a game in G, where the E-associated game $(N, v^*_{\lambda,E})$ is defined by

$$v^*_{\lambda,E}(S) = \begin{cases} 0, & \text{if } S = \emptyset \\ v(S) + \lambda \sum_{j\in N\setminus S} [v(S \cup \{j\}) - v(S) \\ \quad -SC_j(N, v)], & \text{o.w.} \end{cases} \quad (1)$$

Clearly, $v^*_{\lambda,E}$ is a game in G with $v^*_{\lambda,E}(N) = v(N)$ and $v^*_{\lambda,E}(N \setminus \{i\}) = v(N \setminus \{i\})$ for all $i \in N$. Also, if $S \neq \emptyset$ then

$$v^*_{\lambda,E}(S) = \Big(1 - \lambda(n - s)\Big)v(S) + \lambda \sum_{j\in N\setminus S} v(S \cup \{j\}) - \lambda \sum_{j\in N\setminus S} SC_j(v). \quad (2)$$

Hwang [3] showed that the EANSC value is the unique solution satisfying Pareto optimality, symmetry and translation covariance, associated consistency (with respect to the E-associated game) and continuity. Let us first introduce these axioms.

Axiom 1. Pareto Optimality (PO): For all games (N, v) in G,
$$\sum_{j\in N} \sigma_j(N, v) = v(N).$$

Axiom 2. Symmetry (SYM): For all games (N, v) in G, $\sigma_j(N, v) = \sigma_{\pi j}(N, \pi v)$ for all j in N, where π is a permutation on N and $(N, \pi v)$ is defined by $\pi v(\pi S) = v(S)$ for all $S \subseteq N$.

Axiom 3. Translation Covariance (TC): For all games (N, v) in G and for all $x \in \mathbb{R}^N$, $\sigma(N, v) + x = \sigma(N, v + x)$, where $(N, v + x)$ is defined by $(v + x)(S) = v(S) + \sum_{i\in S} x_i$ for all $S \subseteq N$.

Axiom 4. Continuity (CONT): For all converging sequences $\{(N, v_k)\}_{k=1}^{\infty}$ the limit of which is game $(N, \overline{v})$, we have
$$\lim_{k\to\infty} \sigma(N, v_k) = \sigma(N, \overline{v}).$$

Axiom 5. Associated Consistency w.r.t. E-associated game (AC-E): For all games (N, v) and the E-associated game $(N, v^*_{\lambda,E})$, $\sigma(N, v) = \sigma(N, v^*_{\lambda,E})$.

Theorem 1 (*Hwang [3]*). *There is a unique solution on G satisfying PO, SYM, TC, AC-E for* $0 < \lambda < \frac{2}{n-1}$, *and CONT, which is the EANSC value.*

4. Matrix approach to E-associated game

Hwang [3] introduced a sequence of E-associated games as follows: For each game (N, v) in G, the sequence of the E-associated games, $\{(N, v^{m*}_{\lambda,E})\}_{m=0}^{\infty}$, is defined by $v^{0*}_{\lambda,E} = v$, $v^{1*}_{\lambda,E} = v^*_{\lambda,E}$, and $v^{(m+1)*}_{\lambda,E} = (v^{m*}_{\lambda,E})^*_{\lambda,E}$.

By Eq. (2), the term $v^{m*}_{\lambda,E}(S)$ can be expressed as a linear combination of $v(T)$ for all $T \subseteq N$, that is

$$v^{m*}_{\lambda,E}(S) = \sum_{\substack{T \\ T\subseteq N}} \gamma^S_m(T)v(T), \quad (3)$$

where $\gamma^S_m(T) \in \mathbb{R}$ and $\gamma^S_m(\emptyset) = 0$.

The main purpose is to determine the coefficients of the game representation (3) of the m-repeated E-associated game $(N, v^{m*}_{\lambda,E})$. Hwang [3] showed that the coefficients $\gamma^S_m(T)$ satisfy the specified recursive relationships. Also, Hwang [3] showed that the sequence of m-repeated E-associated games $\{(N, v^{m*}_{\lambda,E})\}_{m=0}^{\infty}$ converges and that the limit game is the sum of a constant game and an inessential game.

In this section, we adopt a matrix approach to show the convergence result of the sequence $\{(N, v^{m*}_{\lambda,E})\}_{m=0}^{\infty}$. We define a new type of matrix to apply matrix theory to cooperative game theory as follows:

A game (N, v) is always denoted by its column vector of worths of all coalitions $S \subseteq N$ in the lexicographic order (or in the traditional order, where one-person coalitions are at the top, etc.), i.e. $\vec{v} = (v(S))_{S\subseteq N, S\neq\emptyset}$. If no confusion arises, we write v instead of $\vec{v}$.

Definition 2. A matrix M is called a row (resp. column)-coalitional matrix if its rows (resp. column) are indexed by coalitions $S \subseteq N$ in the lexicographic order. And a row-coalitional matrix $M = [\vec{m}_S]_{S\subseteq N, S\neq\emptyset}$ is row-inessential if the row-vector of M indexed by coalition S verifies $\vec{m}_S = \sum_{i\in S} \vec{m}_i$ for all $S \subseteq N$, where $\vec{m}_i$ denotes the row-vector of M indexed by coalition $\{i\}$.

Definition 3. A row-coalitional matrix $M = [\vec{m}_S]_{S\subseteq N, S\neq\emptyset}$ is row-constant if the row-vector of M verifies $\vec{m}_S = \vec{m}_T$ for all $S, T \subseteq N$.