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Financial contagion and asset liquidation strategies

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ABSTRACT

This paper provides a framework for modeling the financial system with multiple illiquid assets during a crisis. This work generalizes the paper by Amini et al. (2016) by allowing for differing liquidation strategies. The main result is a proof of sufficient conditions for the existence of an equilibrium liquidation strategy with corresponding unique clearing payments and liquidation prices. An algorithm for computing the maximal clearing payments and prices is provided.

1. Introduction

Financial contagion occurs when the distress of one bank jeopardizes the health of other financial firms, and can ultimately spread to the real economy. The spread of defaults in the financial system can occur due to both local connections, e.g., contractual obligations, and global connections, e.g., through the prices of assets due to mark-to-market valuation. As evidenced by the 2007–2009 financial crisis, the cost of a systemic event is tremendous, thus requiring a detailed look at the contributing factors. In this current paper, we will construct and analyze an extension of the financial contagion model of [14] to include multiple illiquid assets with fire sales.

The baseline network model of [14] considers an interbank network of nominal obligations. That paper studies the propagation of defaults through the financial system due to unpaid liabilities. Existence and uniqueness is proven in this base model, as well as algorithms to compute the clearing payments vector which captures the losses in the system. This model has been extended in multiple avenues, including bankruptcy costs, cross-holdings and fire sales. [5] studies these three extensions in a single model; we refer to that work and [25] for a review of the prior literature. [7] studies the effects of liability concentration and network topology on systemic risk via majorization-based tools. Bankruptcy costs have been studied in, e.g., [16,23,15,20,5,7]. Cross-holdings have been studied in, e.g., [16,15,5]. Fire sales for a single (representative) illiquid asset have been studied in, e.g., [10,22,19,2,9,5,3]. For multiple illiquid assets, [12,13] present a framework for modeling and estimating the volatility and correlations of asset prices during a fire sale. In contrast to the present paper, in those publications the financial institutions do not exist within a financial network—by considering the setting as such they are able to study multi-period and continuous-time models, which are not discussed in the scope of the current paper. Similarly, [8] considers a multiasset system in which financial contagion happens solely through balance sheet linkages without a network of interbank liabilities; that paper fixes a specific nonbanking demand to compare different asset allocation strategies. The results of that paper on the robustness of a liquidity-based allocation would also be true in the current model, though the choice of liquidation strategy will result in a modified optimal allocation. A mathematical analysis in this vein is beyond the scope of the current work.

Models of financial contagion and systemic risk have been studied empirically in, e.g., [17,27,11,20]. These studies show that it is unlikely that financial contagion can be captured by the base model of contractual obligations. Thus we extend the network model of [14] to include multiple illiquid assets. We study the case in which a fire sale is triggered if liquid capital (e.g., cash) is insufficient to cover the obligations of a firm, as was studied in, e.g., [3,5]. This is in comparison with the equilibrium model presented in [10] with a single, representative, illiquid asset that is sold if a capital adequacy requirement is violated. The model presented herein is extended in [18] in the direction of [10,8] by explicitly stipulating leverage requirements. We first briefly extend the results from [3] for existence and uniqueness of the clearing payments and equilibrium prices under known liquidation strategies. The main result is to prove existence of a joint clearing payments, asset prices, and an equilibrium liquidation strategy for each financial institution – a game theoretic liquidation strategy –

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and uniqueness of the clearing payments and prices under such a liquidation strategy. We finish by providing, under the necessary conditions, a fictitious default algorithm for computing the maximal clearing payments and prices; we refer to, e.g., [14,23,3] for earlier discussions of this iterative algorithm.

2. Setting

Consider a financial system with n financial institutions (e.g., banks, hedge funds, or pension plans) and a financial market with m illiquid assets. We denote by $p \in \mathbb{R}^n_+$ the realized payments of the banks, $q \in \mathbb{R}^m_+$ the prices of the illiquid assets. There is an additional – liquid – asset in which all liabilities must be paid. Throughout this paper we will use the notation $x \wedge y$ and $x \vee y$ for $x, y \in \mathbb{R}^d$ for some $d \in \mathbb{N}$ to denote

$$x \wedge y = (\min(x_1, y_1), \min(x_2, y_2), \dots, \min(x_d, y_d))^\mathsf{T},$$

$$x \vee y = (\max(x_1, y_1), \max(x_2, y_2), \dots, \max(x_d, y_d))^\mathsf{T}.$$

As described in [14], any financial agent $i \in \{1, 2, \dots, n\}$ may be a creditor or obligor to other agents. Let $\bar{p}_{ij} \geq 0$ be the contractual obligation that firm i owes to firm j. Further, we assume that no firm has an obligation to itself, i.e., $\bar{p}_{ii} = 0$. The *total liabilities* of agent i are given by $\bar{p}_i := \sum_{j=1}^n \bar{p}_{ij}$. We can define the vector $\bar{p} \in \mathbb{R}^n_+$ as the vector of total obligations of each firm. The *relative liabilities* of firm i to firm j, i.e., the fractional amount of total liabilities that firm i owes to firm j, are given by $a_{ij} = \frac{\bar{p}_{ij}}{\bar{p}_i}$ if $\bar{p}_i > 0$ and $a_{ij} \in \mathbb{R}$ arbitrary if $\bar{p}_i = 0$. We define the matrix $A = (a_{ij})_{i,j=1,2,\dots,n}$ with the property $\sum_{j=1}^n a_{ij} = 1$ for any i with $\bar{p}_i > 0$. In the case that $\bar{p}_i = 0$ we are able to choose a_{ij} arbitrarily as it only appears as a multiplier of a variable identically equal to 0. Any financial firm may default on their obligations if sufficient liquid capital is not available. We assume, as per [14], that in case of default the realized payments will be made in proportion to the size of the obligations, i.e., based on the relative liabilities matrix A.

Each firm $i = 1, 2, \dots, n$ has an initial endowment of $x_i \geq 0$ in liquid assets and $s_i \in \mathbb{R}^m_+$ in illiquid assets. That is, agent i holds $s_{ik} \geq 0$ units of illiquid asset $k = 1, 2, \dots, m$. Thus the vector of liquid endowments is given by $x \in \mathbb{R}^n_+$ and the matrix of illiquid endowments is given by $S = (s_{ik})_{i=1,2,\dots,n;\ k=1,2,\dots,m} \in \mathbb{R}^{n\times m}_+$. The price of the illiquid assets is given by a vector valued *inverse demand function* $F : \mathbb{R}^m_+ \to [0, \bar{q}] \subseteq \mathbb{R}^m_+$ for maximum prices $\bar{q}_k$ for asset $k = 1, 2, \dots, m$. Note that we allow for liquidation of one asset to potentially influence the prices of the other assets as well during a fire sale. This would allow us to include correlations of asset prices during fire sales as studied in [12,13]. The inverse demand function maps the quantity of each asset to be sold into a price per share. We will impose the following assumption for the remainder of this paper.

Assumption 2.1. The inverse demand function $F : \mathbb{R}^m_+ \to [0, \bar{q}]$ is continuous and nonincreasing.

In contrast to this setting, [8] utilizes a demand curve for the nonbanking sector rather than an inverse demand function. The results of this paper can be considered in that framework by constructing the equivalent inverse demand function from the nonbanking sector's demand.

We now present a comparable setting to that in [10,3]. We will assume that firms use mark-to-market accounting rules, so that the value of firm i's liquid and illiquid endowment is given by $x_i + q^\mathsf{T} s_i := x_i + \sum_{k=1}^m s_{ik} q_k$ when the vector of prices is given by $q \in \mathbb{R}^m_+$. Additionally, each firm i receives payments from other firms j in proportion to the size of obligations, as described above. That is, firm j will make payment to firm i in the amount of $p_{ji} = a_{ji} p_j$ if firm j pays $p_j \geq 0$ into the system. Thus, the wealth of firm i, taking into account the payments that firm i must make, is given by $x_i + \sum_{k=1}^m s_{ik} q_k + \sum_{j=1}^n a_{ji} p_j - p_i$. By assuming limited liabilities of the firms, i.e., no firm will go into debt to pay its obligations, the wealth of any firm i must be greater than or equal to 0. Thus by rearranging terms we deduce that the payments made by firm i is bounded above by its mark-to-market valuation, i.e., $p_i \leq x_i + \sum_{k=1}^m s_{ik} q_k + \sum_{j=1}^n a_{ji} p_j$. Assuming that firm i must first pay all of its debts before reporting positive wealth, under pricing vector q,

$$p_i = \bar{p}_i \wedge \left(x_i + \sum_{k=1}^m s_{ik} q_k + \sum_{j=1}^n a_{ji} p_j \right).$$

That is, the amount that firm i pays into the financial system is the minimum of its total liabilities $\bar{p}_i$ and the mark-to-market value of its assets $x_i + \sum_{k=1}^m s_{ik} q_k + \sum_{j=1}^n a_{ji} p_j$.

However, it may not be possible for a firm i to pay all obligations $\bar{p}_i$ with liquid holdings $x_i + \sum_{j=1}^n a_{ji} p_j$. This shortfall, $(\bar{p}_i - x_i - \sum_{j=1}^n a_{ji} p_j)^+ := (\bar{p}_i - x_i - \sum_{j=1}^n a_{ji} p_j) \vee 0$, must be made whole, if possible, through the liquidation of assets. Implicitly we assume that a firm will only sell illiquid assets after it has exhausted its store of liquid capital. Due to the price impact (modeled by the inverse demand function F), and the use of mark-to-market accounting, this is the strategy that an equity maximizer would employ. This is in contrast to the work by [10] in which assets are liquidated in order to satisfy a capital adequacy requirement. Unlike in the single illiquid asset case (cf. [5]), we cannot infer more properties without a discussion of the liquidation strategies employed by the financial firms.

3. Clearing mechanism under known liquidation strategy

In this section we consider the realized payments that each firm is able to make under limited liabilities (i.e., no firm pays more than it owes $\bar{p}$) and the realized asset prices after fire sales given a strategy of how the assets are liquidated. That is, we will define the *liquidation function* $\gamma_{ik} : [0, \bar{p}] \times [0, \bar{q}] \to \mathbb{R}_+$ to be the number of units of asset $k = 1, 2, \dots, m$ that firm $i = 1, 2, \dots, n$ wishes to sell. A financial agent will sell assets in order to cover obligations that it cannot meet through its liquid endowment (and realized payments from other firms) alone. For notational simplicity we will say that

$$\gamma_i(p, q) = (\gamma_{i1}(p, q), \gamma_{i2}(p, q), \dots, \gamma_{im}(p, q))^\mathsf{T} \in \mathbb{R}^m_+$$

is the vector of units of illiquid assets which agent i wishes to sell under payments $p \in \mathbb{R}^n_+$ and asset prices $q \in \mathbb{R}^m_+$. Further denote by $\gamma(p, q) \in \mathbb{R}^{n\times m}_+$ to be the matrix of all asset liquidations under payments p and prices q.

We will assume that short-selling is not allowed in the market. Therefore the number of units of asset k that firm i wants to sell, for a fixed payment vector p and price vector q, is given by $s_{ik} \wedge \gamma_{ik}(p, q)$. However, if these sales were actualized, this leads to an updated price $q' \in \mathbb{R}^m_+$ given by the liquidations $s_{ik} \wedge \gamma_{ik}(p, q)$ due to price impact. The updated price is thus given by the inverse demand function, i.e.,

$$q' = F\left(\sum_{i=1}^n [s_i \wedge \gamma_i(p, q)] \right).$$

The goal is to find an equilibrium price vector so that the quoted prices take into account the realized liquidations and vice versa, i.e., $q' = q$.

Due to price impact of selling the illiquid assets, firms will generally want to liquidate the fewest assets necessary under payments $p \in \mathbb{R}^n_+$ and prices $q \in \mathbb{R}^m_+$. As such, we will impose the following *minimal liquidation* condition on the liquidation function γ.