Engineering Analysis with Boundary Elements 37 (2013) 836–841

Contents lists available at SciVerse ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

Degenerate scale for the Laplace problem in the half-plane; Approximate logarithmic capacity for two distant boundaries

A. Corfdir [a,*], G. Bonnet [b]

[a] Université Paris-Est, Laboratoire Navier (UMR 8205), CNRS, ENPC, IFSTTAR, F-77455 Marne-la-Vallée, France
[b] Université Paris Est, Laboratoire Modélisation et Simulation Multi-Echelle, MSME UMR 8208 CNRS, 5 boulevard Descartes, 77454 Marne la Vallée Cedex, France

ARTICLE INFO

Article history:
Received 12 November 2012
Accepted 26 February 2013
Available online 28 March 2013

Keywords:
Laplace equation
Boundary element method
Plane problems
Green's function
Degenerate scale
Half-plane
Exterior problem

ABSTRACT

We study the problem of finding a degenerate scale for Laplace equation in a half-plane. It is shown that if the boundary condition on the line bounding the half-plane is of Dirichlet type, there is no degenerate scale. In the case of a boundary condition of Neumann type, there is a degenerate scale, which is shown to be the same as the one for the symmetrized contour with respect to the boundary line in the full plane. We show next a formula for obtaining the degenerate scale of a domain made of two parts, when the components are far from each other, which allows to obtain the degenerate scale for the symmetrized contour. Finally, we give some examples of evaluation of the degenerate scale both by an approximate formula and by a numeric evaluation using integral methods. These evaluations show that the approximate solution is still valid for small values of the distance between symmetrized contours.

1. Introduction

One considers for a given contour S all contours obtained by any scaling of S. Early results by Jaswon [21], based on the work of Muskhelishvili [23], showed that for any boundary S of a plane domain, there is one and only one degenerate scale which leads to the non-invertibility of the integral operator: $q \mapsto \psi_q(x) = -\int_S \ln\|x-y\| q(y)\, dS_y$. More precisely, it can be shown that there is a distribution $q(x)$ such that the application $\psi_q(x)$ is null for any x. Hayes and Kellner [17] have shown using complex variables that a degenerate scale is reached when the "transfinite diameter" of the outer domain, defined from a conformal mapping of the domain from the outer unit circle, is equal to 1. From another point of view, the transfinite diameter defined in the complex plane is equal to the logarithmic capacity as shown by Hille [19]. The logarithmic capacity, whose definition is recalled in Section 5, is defined without using a conformal mapping, which makes it easier to handle. Yan and Sloan [29] give a review of the main properties of the logarithmic capacity. It is possible to derive analytically the logarithmic capacity of some domains by using conformal mapping, for example for regular N-gon domains [22]. The numerical calculation of the logarithmic capacity has been studied by Dijkstra and Hochstenbach [16] and Chen et al. [5].

The degenerate scale problem is still investigated by different authors: Yan and Sloan [29] have studied the case of an open boundary, Coscia and Russo [14] have studied the case of a Lipschitz boundary. The problem has been extended to multiply connected domains [6,8]. Many authors have discussed how to eliminate this problem by scaling [12], by adding a constant to the fundamental solution [12,5], by adding an unknown and an additional equation [11,20], by adding an additional collocation point [4] or by using an hypersingular formulation as noted by Chen et al. [5]. Practically, a 2D problem can be very often considered as an approximation of a 3D problem at the vicinity of a finite line source. In this case, a specific kernel depending of the geometry of the 3D problem can be defined [1]. This kernel ensures that the scale of the 2D problem is smaller than the degenerate scale if a condition ensuring that 2D modeling is well adapted to the original 3D problem is satisfied. A more extended review on the problem of degenerate scale for Laplace equation can be found in Chen [3]. More generally, some authors have studied the degenerate scale for other equations having a logarithmic term in the Green function: elasticity in the plane [18,13,26,27,9,10], biharmonic equation in the plane [15,6]. These equations exhibit similar though more complicated behaviors, with generally several degenerate scales.

This paper is devoted to the study of degenerate scales for problems in a half-plane, which seems, from our knowledge, not having been studied with as much extent as the problem in a full plane. Elementary solutions of the Laplace equation for the half-

* Corresponding author. Tel.: +33 164153521.
E-mail addresses: corfdir@cermes.enpc.fr (A. Corfdir), Guy.Bonnet@univ-mlv.fr (G. Bonnet).

0955-7997/$ - see front matter
http://dx.doi.org/10.1016/j.enganabound.2013.02.009

plane is based on the method of images. This method has been found very early by Thomson [25] and is still the object of research (see e.g. [8,7]). In this paper, we use the results established for the plane and draw next basic results for the case of the half-plane, either for Dirichlet or Neumann boundary conditions at the boundary of the half-plane. An approximate of the degenerate scale for the exterior problem is given, when the diameter of the inner boundary is small compared with the distance between this inner boundary and the boundary line of the half-plane.

2. Relation between the degenerate scale in a half-plane and an associated problem in the full plane

We first recall the link between the loss of unicity of the integral boundary equation and the existence of a non-trivial solution of the equation

$$\int_S G(x-y)q(y)\,\mathrm{d}S_y = 0, \quad x \in S \tag{1}$$

where $G(x,y) = -\ln\|x-y\|/2\pi$. Assuming the loss of unicity for Dirichlet condition, making the difference between two solutions, we get a function $u(x)$ with $q(x) = \partial u/\partial n$ satisfying the following boundary equation, $\frac{1}{2}u(x) + \int_S (\partial G/\partial n)(x-y)u(y)\,\mathrm{d}S_y = \int_S G(x-y)q(y)\,\mathrm{d}S_y$ with $u(x) = 0$ for $x \in S$ and q being a non-null function on S. The boundary integral equation then gives $\int_S G(x-y)q(y)\,\mathrm{d}S_y = 0$ for $x \in S$. Conversely, if there exists q a non-trivial solution of (1), then the function $u(x) = \int_S G(x-y)q(y)\,\mathrm{d}S_y$ is a non-null solution of the Laplace equation which vanishes on S.

The above argument for the plane can be applied for the half-plane using the appropriate kernel. Then, the degenerate scale in the half-plane is related to the non-invertibility of the operator $q \mapsto \psi_q(x) = -\int_S G_\alpha(x-y)q(y)\,\mathrm{d}S_y$ where G_α is the Green's function associated with conditions applied to the boundary line of the half-plane. We can consider two different problems in the half-plane according to the type of the condition at the line Δ bounding the half-plane: Neumann condition if the normal derivative is null on Δ or Dirichlet conditions if the function is null on Δ (Fig. 1).

We consider the standard Green solution for Laplace equation in the plane: $G(x,y) = -\ln\|x-y\|/2\pi$. For the half-plane, the image method gives the following Green function:

$$G_\alpha(x,y) = G(x,y) + \epsilon G(\overline{x},y) \tag{2}$$

where $\overline{x}$ is the image of x, $\alpha = N$ and $\epsilon = 1$ for Neumann condition and $\alpha = D$ and $\epsilon = -1$ for Dirichlet condition on line Δ. Considering that many results are found in the literature on the degenerate scale in the plane, we intend to build a relation between the degenerate scale in the half-plane for a given boundary and the degenerate scale for an associate problem in the full plane.

Theorem 1. *If a problem in the half-plane is at a degenerate scale, then an associate problem can be built, which is at a degenerate scale in the full plane.*

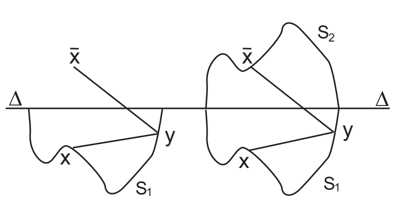

Fig. 1. The image method (left) and the symmetrization of the boundary (right).

Proof. We extend the function q defined on S_1 to a function q_α defined on $S_1 \cup S_2$ in the following way:

$$q_\alpha(y) = \begin{cases} q(y) & \text{if } y \in S_1 \\ \epsilon q(\overline{y}) & \text{if } \overline{y} \in S_2 \end{cases} \tag{3}$$

with $\alpha = N$ and $\epsilon = 1$ for Neumann condition and $\alpha = D$ and $\epsilon = -1$ for Dirichlet condition on line Δ. Using $G(\overline{x},y) = G(x,\overline{y})$, we can write

$$\begin{aligned}\int_{S_1} G_\alpha(x,y)q(y)\,\mathrm{d}S_y &= \int_{S_1} (G(x,y) + \epsilon G(\overline{x},y))q(y)\,\mathrm{d}S_y \\ &= \int_{S_1} G(x,y)q(y)\,\mathrm{d}S_y + \int_{S_1} G(x,\overline{y})\epsilon q(y)\,\mathrm{d}S_y \\ &= \int_{S_1} G(x,y)q(y)\,\mathrm{d}S_y + \int_{S_2} G(x,y)q_\alpha(y)\,\mathrm{d}S_y \\ &= \int_{S_1 \cup S_2} G(x,y)q_\alpha(y)\,\mathrm{d}S_y \end{aligned} \tag{4}$$

If the domain S_1 is at a degenerate scale, the integral operator is not invertible for the problem in the half-plane with Neumann or Dirichlet condition on the Δ line; therefore, there is a non-null function q such that: $\int_{S_1} G_\alpha(x,y)q(y)\,\mathrm{d}S_y = 0$.

Then, the extended function q_α is such (4) that $\int_{S_1 \cup S_2} G(x,y)q_\alpha(y)\,\mathrm{d}S_y = 0$.

Hence $S_1 \cup S_2$ is at the degenerate scale for the problem in the plane. This proves Theorem 1. □

This result is particularly useful, because it allows to apply all results obtained in the plane for the symmetrized domain $S_1 \cup S_2$.

Before to use this associate domain in Section 5, two general results on the degenerate problems associated with Dirichlet or Neumann boundary conditions will be established.

3. The Laplace problem in the half-plane with a Dirichlet condition on the boundary line

In this section, we assume that a null Dirichlet condition is applied on the boundary line. In this case, we establish the following theorem:

Theorem 2. *There is no degenerate scale for the Laplace problem in the half-plane with Dirichlet condition on the boundary line.*

Proof. AssumebycontradictionthattheboundaryS_1isatthedegeneratescalefortheLaplaceprobleminthehalf-planewithDirichletconditionontheline boundingthehalf-plane. Thereexistsanon-nullfunction$q$suchthat: $\int_{S_1} G_D(x,y)q(y)\,\mathrm{d}S_y = 0$.

If we change the scale of the problem by a factor λ (Fig. 2), we define

$$q_\lambda(y) = q\left(\frac{y}{\lambda}\right) \tag{5}$$

From the definition of $G_D(x,y)$, we get $G_D(\lambda x,\lambda y) = G_D(x,y)$. Hence, we can write

$$\int_{\lambda S_1} G_D(x,y)q_\lambda(y)\,\mathrm{d}S_y = \lambda \int_{S_1} G_D(\lambda x,\lambda y)q(y)\,\mathrm{d}S_y$$

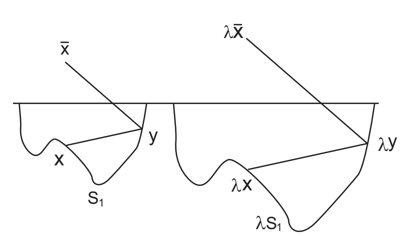

Fig. 2. Change of scale.