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A discrete variant of Farkas' Lemma

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ABSTRACT

We report a discrete variant of Farkas' Lemma in the setting of a module over a linearly ordered commutative ring. The ring may contain zero divisors, and need not be associative nor unital, but we need a certain hypothesis about the ring. Finally, we discuss the result and compare it with other related results found in the literature.

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1. Introduction

Farkas' Lemma [10] proved to be useful in the duality theory of mathematical optimization. The original result has been extended and generalized in various ways, see [8,16–18,24,9] for a recent survey. A particular generalization of Farkas' Lemma, due to Bartl, is in the setting of a vector space (of finite or infinite dimension) over a linearly ordered (commutative or skew) field [1]; shorter algebraic proofs of this result can be found in [2–4]. The algebraic nature of the "very short" proof [4] inspired us to develop a discrete variant of the result: we now consider Farkas' Lemma in the setting of a module over a linearly ordered commutative ring.

To illuminate our motivation, let R be a linearly ordered ring, such as the ring $R = \mathbb{Z}$ of the integer numbers. Let $\boldsymbol{A} \in R^{m\times n}$ be a matrix and let $\boldsymbol{c} \in R^n$ be a vector. Roughly, assuming the Farkas condition that $\boldsymbol{Ax} \le \boldsymbol{o}_m$ implies $\boldsymbol{c}^T\boldsymbol{x} \le 0$ for all $\boldsymbol{x} \in R^n$, we ask whether there exists a non-negative $\boldsymbol{u} \in R^m$ such that $\boldsymbol{A}^T\boldsymbol{u} = \boldsymbol{c}$. Although we do not answer the question in full—we simplify it, we conclude that there exist a non-negative $\boldsymbol{u} \in R^m$ and a *positive* $r \in R$ such that $\boldsymbol{A}^T\boldsymbol{u} = \boldsymbol{c}r$, i.e., the vector $\boldsymbol{c}$ also gets scaled to its "proper length"—we still go further in at least two respects: First, the ring R can be any arbitrary linearly ordered commutative ring; the ring need not be associative, need not be unital, and, namely, can contain zero divisors. We shall, however, assume a certain hypothesis about the ring R that is explained at the beginning of Section 3. Second, we consider the Farkas condition in the setting of an arbitrary module W over the ring R, not only in the free module of the form $W = R^n$ as we did above.

2. Notation

In this section, we recall the notation we use to present our result.

The symbol R denotes a commutative ring and V denotes a module over the ring R. We assume that a binary relation "$\le$" and "$\preceq$" is given on the ring R and the module V, respectively. We say that V is a *linearly ordered module* over the *linearly ordered ring* R iff, for all $\lambda, \mu \in R$ and for all $u, v \in V$, it holds

$$\begin{array}{cc}
\lambda \ge 0 \vee \lambda \le 0, & u \succeq 0 \vee u \preceq 0, \\
\lambda \ge 0 \wedge \lambda \le 0 \Longrightarrow \lambda = 0, & u \succeq 0 \wedge u \preceq 0 \Longrightarrow u = 0, \\
\lambda \ge 0 \wedge \mu \ge 0 \Longrightarrow \lambda + \mu \ge 0, & u \succeq 0 \wedge v \succeq 0 \Longrightarrow u + v \succeq 0, \\
\lambda \ge 0 \wedge \mu \ge 0 \Longrightarrow \lambda\mu \ge 0, & \lambda \ge 0 \wedge u \succeq 0 \Longrightarrow \lambda u \succeq 0, \\
\lambda \le \mu \iff \lambda - \mu \le 0, & u \preceq v \iff u - v \preceq 0,
\end{array}$$

where we have used the usual convention that $\lambda \ge \mu$ or $u \succeq v$ iff $\mu \le \lambda$ or $v \preceq u$, respectively. An element $\lambda \in R$ is *positive*, *negative*, *non-negative*, *non-positive*, and *non-zero*, iff $\lambda > 0$, $\lambda < 0$, $\lambda \ge 0$, $\lambda \le 0$, and $\lambda \ne 0$, respectively. These five concepts are defined for elements of the module V analogously.

For a vector $u \in V$ and for a scalar $\lambda \in R$, we let $\iota u\lambda = \lambda u$, the λ-multiple of the vector u. That is, the symbol "ι" (Greek letter iota) means the next two elements of the module and ring are to be transposed and multiplied in the new order. Notice that, for a fixed $u \in V$, we thus obtain a linear mapping or homomorphism $\iota u\colon R \to V$ with $\iota u\colon \lambda \mapsto \lambda u$.

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Let W be another module over the commutative ring R. Given a linear form $\alpha: W \to R$ and a vector $u \in V$, we can compose the form α with $\iota u: R \to V$. We denote the composition by $\iota u\alpha$, and we have $\iota u\alpha x = (\alpha x)u$ for any $x \in W$. Consider a linear mapping or homomorphism $\gamma: W \to V$. Given a constant $r \in R$, then $r\gamma$ is the r-multiple of the mapping γ. We have $r\gamma x = r(\gamma x)$ for any $x \in W$. Notice that $r\gamma$ is a linear mapping, because the ring R is commutative, and that we drop parentheses around the argument "x" if they are unnecessary. That is $\alpha x = \alpha(x)$ and $\gamma x = \gamma(x)$ for all $x \in W$.

For a non-negative natural number m, let R^m and V^m be endowed with the respective natural structure of a module over the ring R. We stipulate that the elements $\boldsymbol{\lambda} \in R^m$ and $\boldsymbol{u} \in V^m$ are columns of their components $\lambda_1, \dots, \lambda_m \in R$ and $u_1, \dots, u_m \in V$, respectively. The superscript "T" denotes transposition so that $\boldsymbol{u}^T$ is a row. We then have $\iota\boldsymbol{u}^T\boldsymbol{\lambda} = \iota u_1\lambda_1 + \cdots + \iota u_m\lambda_m = \lambda_1 u_1 + \cdots + \lambda_m u_m$ as usual. Analogously, a linear mapping $A: W \to R^m$ consists of the respective linear forms $\alpha_1, \dots, \alpha_m: W \to R$ arranged in a column. Notice that a fixed $\boldsymbol{u} \in V^m$ induces a linear mapping $\iota\boldsymbol{u}^T: R^m \to V$ with $\iota\boldsymbol{u}^T: \boldsymbol{\lambda} \mapsto \iota\boldsymbol{u}^T\boldsymbol{\lambda}$ for any $\boldsymbol{\lambda} \in R^m$. The composition of both mappings is denoted $\iota\boldsymbol{u}^T A$. We have $\iota\boldsymbol{u}^T Ax = (\alpha_1 x)u_1 + \cdots + (\alpha_m x)u_m$ for all $x \in W$.

3. A discrete variant of Farkas' Lemma

In this section, we adapt the idea of Bartl's "very short" algebraic proof [4] of Farkas' Lemma to establish a discrete variant of it.

Let R be a linearly ordered commutative ring. The ring R need not be associative and need not be unital (i.e., need not possess the unit element, neutral with respect to multiplication). An element $a \in R$ is a *zero divisor* iff $a \neq 0$ and there exists a non-zero $b \in R$ such that $ab = ba = 0$. It follows the element b is a zero divisor too. Actually, if $b \geq a$, then all the elements $c \in R$ such that $b \geq c > 0$ are zero divisors. We note, if $b > 0$ is not a zero divisor, then no element $c \in R$ such that $c > b$ is a zero divisor either. Given two positive elements $a, e \in R$, the element $a \in R$ is *infinitely less* than e iff $a < e$ and $ta < e$ for all $t \in R$.

We assume the following properties of the linearly ordered commutative ring R: the ring may contain zero divisors, the ring need not be associative nor unital, but there must exist a positive element $e \in R$ such that $ee > 0$ and, for any element $a \in R$, if $e > a > 0$ and a is a zero divisor, then a is not infinitely less than e.

Remark 1. The element e itself may be a zero divisor, although $ee > 0$. If the ring R is associative and e is of the above property, then e is not a zero divisor; actually, the ring R does not contain any zero divisor at all. The associativity of the ring R is not assumed, however. If there is no zero divisor in the ring, then e can be any positive element and its existence means the ring R is non-trivial.

Furthermore, let V be a linearly ordered module over the linearly ordered ring R. Although the ring R need not be associative, we do assume that $(\lambda\mu)u = \lambda(\mu u)$ for all $\lambda, \mu \in R$ and for all $u \in V$.

Finally, let W be a module over the ring R with $(\lambda\mu)x = \lambda(\mu x)$ for all $\lambda, \mu \in R$ and for all $x \in W$. Last but not least, let $\alpha_1, \dots, \alpha_m: W \to R$ be linear forms, where m is a non-negative natural number, and let $\gamma: W \to V$ be a linear mapping.

Lemma 1 (*A Discrete Variant of Farkas' Lemma*)**.** *Under the above assumptions, the next statement* (A) *implies the subsequent statement* (B)*:*

(A) *For each $x \in W$, it holds*

$$(\alpha_1 x \leq 0 \wedge \cdots \wedge \alpha_m x \leq 0) \Longrightarrow \gamma x \preceq 0. \tag{1}$$

(B) *There exist an $r \in R$ and $u_1, \dots, u_m \in V$ such that $r > 0$ and $u_1, \dots, u_m \succeq 0$, and*

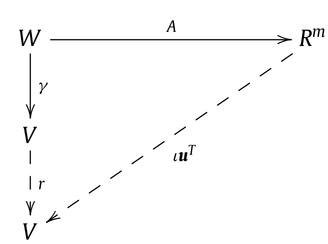
$$r\gamma = \iota u_1\alpha_1 + \cdots + \iota u_m\alpha_m. \tag{2}$$

Remark 2. If $m = 0$, then the empty logical conjunction "$\alpha_1 x \leq 0 \wedge \cdots \wedge \alpha_m x \leq 0$" is logically true by convention. It follows then that $\gamma = o$, the zero mapping $o: W \to V$. Next, the empty logical conjunction "$u_1, \dots, u_m \succeq 0$" is logically true as well and the empty sum "$\iota u_1\alpha_1 + \cdots + \iota u_m\alpha_m$" equals the zero mapping $o: W \to V$ by the respective conventions if $m = 0$.

Remark 3. Statement (A) implies statement (B). The converse implication is not true in general. Recall the module V is *R-torsion free* iff, for all $r \in R$ and for all $u \in V$, we have $ru \neq 0$ whenever both $r \neq 0$ and $u \neq 0$ and r is not a zero divisor. If statement (B) holds, the positive element r is not a zero divisor and the module V is R-torsion free, then statement (A) holds too: If $\alpha_1 x, \dots, \alpha_m x \leq 0$, then $r\gamma x = \iota u_1\alpha_1 x + \cdots + \iota u_m\alpha_m x \preceq 0$. Should we have $\gamma x \succ 0$, then $r\gamma x \succ 0$, a contradiction.

Remark 4. Notice that, if statement (B) holds and the element e is not a zero divisor, then we can assume wlog that the positive element r is not a zero divisor either. Indeed, assume that $r > 0$ and $u_1, \dots, u_m \succeq 0$ are such that (2) holds. If r is a zero divisor, then it holds $e > r > 0$ because e is not a zero divisor. Since r is not infinitely less than e by assumption, there exists a positive $t \in R$ such that $tr \geq e$. We then obtain $t(r\gamma) = t(\iota u_1\alpha_1) + \cdots + t(\iota u_m\alpha_m)$. Since $(\lambda\mu)u = \lambda(\mu u)$ for all λ, μ, u, and R is commutative, Eq. (2) is satisfied with $r := tr$ and $u_i := tu_i$ for $i = 1, \dots, m$ as well.

Remark 5. Given the homomorphisms $\gamma: W \to V$ and $A: W \to R^m$ of the linear forms $\alpha_1, \dots, \alpha_m: W \to R$, statement (B) says that there exist homomorphisms $\iota\boldsymbol{u}^T: R^m \to V$ and $r: V \to V$, with $r: u \mapsto ru$ for any $u \in V$, which make the following diagram commute:



Proof. We prove the assertion of Lemma 1 by induction. If $m = 0$, putting $r = e$, say, statement (B) is obvious by Remark 2. Assume that the assertion has been proved for a non-negative natural number m. We shall prove it for $m + 1$. Thus, assume that it holds

$$\forall x \in W\colon\ (\alpha_1 x \leq 0 \wedge \cdots \wedge \alpha_m x \leq 0 \wedge \alpha_{m+1} x \leq 0) \Longrightarrow \gamma x \preceq 0. \tag{3}$$

If it holds even

$$\forall x \in W\colon\ (\alpha_1 x \leq 0 \wedge \cdots \wedge \alpha_m x \leq 0) \Longrightarrow \gamma x \preceq 0, \tag{4}$$

then we are done: there exist an $r > 0$ and non-negative $u_1, \dots, u_m \in V$ such that $r\gamma = \iota u_1\alpha_1 + \cdots + \iota u_m\alpha_m$ by the induction hypothesis; put $u_{m+1} = 0$. Assume now that (4) does not hold. Then

$$\exists x_{m+1} \in W\colon\ (\alpha_1 x_{m+1} \leq 0 \wedge \cdots \wedge \alpha_m x_{m+1} \leq 0) \wedge \gamma x_{m+1} \succ 0.$$

Since (3) holds, it follows $\alpha_{m+1}x_{m+1} > 0$. Distinguish three cases. First, if $\alpha_{m+1}x_{m+1}$ is not a zero divisor, then $(\alpha_{m+1}x_{m+1})r' > 0$ for any positive $r' \in R$. Second, if $\alpha_{m+1}x_{m+1}$ is a zero divisor and $\alpha_{m+1}x_{m+1} \geq e$, then $(\alpha_{m+1}x_{m+1})r' > 0$ for any positive $r' \in R$

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