Operations Research Letters 45 (2017) 1–5

Contents lists available at ScienceDirect

Operations Research Letters

journal homepage: www.elsevier.com/locate/orl

A single-level reformulation of mixed integer bilevel programming problems

Chuanjia Li [a], Lei Guo [b,*]

[a] Antai College of Economics and Management, Shanghai Jiao Tong University, Shanghai 200030, China
[b] Sino-US Global Logistics Institute, Shanghai Jiao Tong University, Shanghai 200030, China

ARTICLE INFO

Article history:
Received 10 February 2016
Received in revised form
22 October 2016
Accepted 23 October 2016
Available online 5 November 2016

Keywords:
Mixed integer bilevel program
Mixed integer program
Mathematical program with complementarity constraints

ABSTRACT

This paper focuses on the single-level reformulation of mixed integer bilevel programming problems (MIBLPP). Due to the existence of lower-level integer variables, the popular approaches in the literature such as the first-order approach are not applicable to the MIBLPP. In this paper, we reformulate the MIBLPP as a mixed integer mathematical program with complementarity constraints (MIMPCC) by separating the lower-level continuous and integer variables. In particular, we show that global and local minimizers of the MIBLPP correspond to those of the MIMPCC respectively under suitable conditions.

1. Introduction

In this paper, we consider the mixed integer bilevel programming problem (MIBLPP) of the form

$$\begin{array}{ll} \min\limits_{x,y,z} & F(x,y,z) \\ \text{s.t.} & x \in X(y,z), \\ & (y,z) \in S(x), \end{array} \tag{1}$$

where some of the upper-level decision variables x are integer-valued, $F : \Re^{n_1+n_2+n_3} \to \Re$, $X(y,z)$ is a nonempty closed subset in $\Re^{n_1}$ for any $(y,z) \in \Re^{n_2+n_3}$, and $S(x)$ is the set of global minimizers for the lower-level problem parameterized by x

$$\begin{array}{ll} \min\limits_{y,z} & f(x,y,z) \\ \text{s.t.} & (y,z) \in \Xi(x), \end{array} \tag{2}$$

where y represents the lower-level continuous variables, z represents the lower-level integer variables, $f : \Re^{n_1+n_2+n_3} \to \Re$ is a continuously differentiable function with respect to y, and $\Xi(x)$ is defined as

$$\Xi(x) := \left\{(y,z) \in \Re^{n_2} \times Z : g(x,y,z) \le 0, h(x,y,z) = 0\right\}$$

with a set of integers Z in $\Re^{n_3}$ and continuously differentiable functions $g : \Re^{n_1+n_2+n_3} \to \Re^p$, $h : \Re^{n_1+n_2+n_3} \to \Re^q$ with respect to y. Throughout this paper, we assume that the cardinality $|Z|$ of Z is finite and $n_2 > 0$.

Bilevel programming problems have been intensively investigated; see, e.g., the review articles [6,7]. Almost all investigations in the literature are devoted to problems with only continuous variables in the lower-level problem for which the popular way is to reformulate it as a single-level problem, although there exist many practical problems with integer variables in the lower-level problem; see, e.g., [2,4,9,18]. There have also been some progresses made on the numerical methods for solving linear MIBLPPs; see, e.g, [2,3,17,20,23,24].

It is desirable to know whether there exists a minimizer before solving the MIBLPP. Vicente et al. [22] studied the existence of minimizers of linear MIBLPP for different cases corresponding to particularizations of the upper-level and lower-level variables. Unfortunately, for the case where there exist lower-level integer variables and upper-level joint constraints, it is a difficult task to give the existence of minimizers. In this paper, we assume that a minimizer exists for the MIBLPP and focus on deriving its single-level reformulation.

To the best of our knowledge, there are only a few publications on developing numerical methods for solving MIBLPPs with nonlinear functions [10,12,16]. Our proposed single-level reformulation in this paper makes it possible to develop more approaches for solving MIBLPPs as what has been done for bilevel programming problems with only continuous variables in the lower-level problem.

* Corresponding author.
E-mail address: guolayne@sjtu.edu.cn (L. Guo).

http://dx.doi.org/10.1016/j.orl.2016.10.013
0167-6377/

When there is no lower-level integer variable ($n_3 = 0$), we may replace the lower-level problem of the MIBLPP with its Karush–Kuhn–Tucker (KKT) conditions, resulting in a mathematical program with complementarity constraints (MPCC). Their relations in the sense of global and local minimizers have been established in [8] provided that the lower-level problem is convex and Slater's constraint qualification is satisfied. When there are lower-level integer variables ($n_3 > 0$), the lower-level problem is clearly nonconvex and thus we cannot replace the lower-level problem with its KKT conditions directly. In this paper, we reformulate the MIBLPP as a mixed integer mathematical program with complementarity constraints (MIMPCC) by separating the lower-level continuous and integer variables. In order to investigate the relations between the MIBLPP and the MIMPCC, we assume that for any lower-level integer variables, the lower-level problem is solvable and convex with respect to the continuous variables. We show that the global minimizers of the MIBLPP correspond to those of the MIMPCC provided that for any lower-level integer variables, the lower-level problem satisfies the generalized Slater's constraint qualification with respect to the continuous variables. Since bilevel programming is generally nonconvex, it is difficult to find a global minimizer and in some cases, one needs to be happy with obtaining a local minimizer. Thus, it is also necessary to investigate the relations between the local minimizers of the MIBLPP and the MIMPCC. The stronger Slater's constraint qualification and the so-called restricted inf-compactness condition are required to ensure that the local minimizers of the MIBLPP correspond to those of the MIMPCC.

The rest of this paper is organized as follows. In Section 2 we give some background materials and preliminary results. In Section 3 we reformulate the MIBLPP as a single-level MIMPCC. In particular, we show that the global and local minimizers of the MIBLPP correspond to those of the MIMPCC respectively under some suitable conditions. Section 4 concludes the paper.

Notation: For any vectors $a, b \in \Re^n$, $a \perp b$ means that vector a and vector b are perpendicular. Given a point $x \in \Re^n$ and a function $\varphi : \Re^n \to \Re^m$, $\nabla\varphi(x)$ denotes the transposed Jacobian of φ at x.

2. Preliminaries and preliminary results

We first review some constraint qualifications for convex constrained sets. Let $\mathcal{X}$ be a convex constrained set defined by

$$\mathcal{X} := \left\{x : \phi(x) \le 0,\ \varphi(x) = 0\right\}$$

with a convex and continuously differentiable function $\phi : \Re^n \to \Re^l$ and a linear function $\varphi : \Re^n \to \Re^d$. The following two constraint qualifications will be useful in the next section.

Definition 1. (i) We say that Slater's constraint qualification (SCQ) holds for $\mathcal{X}$ if $\nabla\varphi$ has the full column rank and there exists $x_0 \in \Re^n$ such that

$$\phi(x_0) < 0, \qquad \varphi(x_0) = 0.$$

(ii) We say that the generalized Slater's constraint qualification (GSCQ) holds for $\mathcal{X}$ if there exists $x_0 \in \Re^n$ such that

$$\phi(x_0) < 0, \qquad \varphi(x_0) = 0.$$

It is obvious that GSCQ is strictly weaker than SCQ. For convex programming problems, the global minimizers correspond to the associated KKT points under GSCQ. Moreover, it is well-known that SCQ for $\mathcal{X}$ is equivalent to Mangasarian–Fromovitz constraint qualification (MFCQ) at any $x \in \mathcal{X}$: $\nabla\varphi(x)$ has the full column rank and there exists d such that

$$\nabla\phi_i(x)^T d < 0\ i \in \mathcal{I}_\phi(x), \qquad \nabla\varphi(x)^T d = 0,$$

where $\mathcal{I}_\phi(x) := \{i : \phi_i(x) = 0\}$. It also should be noted that GSCQ for $\mathcal{X}$ is equivalent to constant rank MFCQ (CRMFCQ) introduced in [15] at any $x \in \mathcal{X}$: There exists d such that

$$\nabla\phi_i(x)^T d < 0\ i \in \mathcal{I}_\phi(x), \qquad \nabla\varphi(x)^T d = 0.$$

It then follows that $\mathcal{X}$ admits a local error bound at any $x \in \mathcal{X}$ if GSCQ is satisfied; see, e.g., [14, Corollary 4.1].

We next recall some stability results for the parametric nonlinear programming problem

$$NLP(p) \quad \min_x \ \theta(x,p) \quad \text{s.t.} \quad \phi(x,p) \le 0, \quad \varphi(x,p) = 0$$

with functions $\theta : \Re^{n+m} \to \Re$, $\phi : \Re^{n+m} \to \Re^l$ and $\varphi : \Re^{n+m} \to \Re^d$. We denote the feasible region of $NLP(p)$ as

$$\mathbb{X}(p) := \{x :\ \phi(x,p) \le 0,\ \varphi(x,p) = 0\},$$

and the optimal value function as

$$\mathbb{V}(p) := \inf\{\theta(x,p) :\ x \in \mathbb{X}(p)\}.$$

The following restricted inf-compactness condition introduced in [5] is a weak condition ensuring the lower semi-continuity of optimal value function. The popular inf-compactness condition [21] and uniform compactness [11] in the literature are both strictly stronger than restricted inf-compactness; see [13, Pages 1224–1225] for the detailed discussions.

Definition 2 (*[5, Hypothesis 6.5.1]*)**.** We say that restricted inf-compactness holds at $\bar{p}$ if $\mathbb{V}(\bar{p})$ is finite and there exist a compact set Ω and a positive number ϵ_0 such that for all $p \in \mathcal{B}_{\epsilon_0}(\bar{p})$ satisfying $\mathbb{V}(p) < \mathbb{V}(\bar{p}) + \epsilon_0$, $NLP(p)$ has a solution in Ω.

Proposition 1. *Let $\phi(\cdot,\bar{p})$ be convex and continuously differentiable, and $\varphi(\cdot,\bar{p})$ be linear. If the restricted inf-compactness holds at $\bar{p}$ and SCQ holds for $\mathbb{X}(\bar{p})$, then $\mathbb{V}$ is continuous at $\bar{p}$.*

Proof. The restricted inf-compactness implies that $\mathbb{V}$ is lower semi-continuous at $\bar{p}$; see, e.g., [5, Page 246]. Let $\bar{x}$ be a minimizer of $NLP(\bar{p})$. Since SCQ implies that MFCQ holds at $\bar{x}$, it then follows from [11, Theorem 3.3] that $\mathbb{V}$ is upper semi-continuous at $\bar{p}$. The desired result follows immediately. □

Before ending this section, we point out that an MPCC is equivalent to a mixed integer program by introducing binary variables to replace the complementarity constraints when the complementarity functions are continuous and the feasible region is compact [1].

3. Single-level reformulation

This section focuses on the single-level reformulation of the MIBLPP. Let Assumption 1 hold throughout this section.

Assumption 1. Assume that for any $(x,z) \in \Re^{n_1} \times Z$, $f(x,\cdot,z)$ and $g(x,\cdot,z)$ are convex and continuously differentiable functions, and $h(x,\cdot,z)$ is a linear function.

In order to give a single-level reformulation of the MIBLPP, we need use a system of equalities and inequalities to characterize the optimal solution set $S(x)$ of problem (2). Although problem (2) is a nonconvex problem, we observe that for any $(x,z) \in \Re^{n_1} \times Z$, the problem

$$P_z(x):\ \min_y\left\{f(x,y,z) :\ (y,z) \in \Xi(x)\right\}$$

is a convex problem under Assumption 1 and hence it is possible to replace the lower-level problem with some KKT conditions provided a certain regularity condition is satisfied. The following two examples may give us some insights on how to characterize $S(x)$ by separating the lower-level integer and continuous variables.