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Small-noise limit of the quasi-Gaussian log-normal HJM model

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ABSTRACT

Quasi-Gaussian HJM models are a popular approach for modeling the dynamics of the yield curve. This is due to their low dimensional Markovian representation, which greatly simplifies their numerical implementation. We present a qualitative study of the solutions of the quasi-Gaussian log-normal HJM model. Using a small-noise deterministic limit we show that the short rate may explode to infinity in finite time. This implies the explosion of the Eurodollar futures prices in this model. We derive explicit explosion criteria under mild assumptions on the shape of the yield curve.

1. Introduction

HJM models [10] are widely used in financial practice for modeling fixed income, credit and commodity markets [1]. These models specify the dynamics of the yield curve $f(t,T)$ as

$$df(t,T) = \sigma_f(t,T)^T dW(t) + \sigma_f^T(t,T)\left(\int_t^T \sigma_f(t,s)ds\right)dt, \qquad (1)$$

where $W(t)$ is a vector Brownian motion under the risk-neutral measure $\mathbb{Q}$ and $\{\sigma_f(t,T)\}_{t\le T}$ is a family of vector processes. The numerical simulation of these models is complicated by the fact that the entire yield curve $f(t,T)$ has to be simulated. Lattice and tree simulation methods require an exponentially large number of nodes. For this reason the simulation of these models is restricted in practice to Monte Carlo methods.

The quasi-Gaussian HJM models [1–3,5,14] were introduced to simplify the simulation of the HJM models. They are obtained by assuming a separable form for the volatility $\sigma_f(t,T)^T = g(T)^T h(t)$ where g is a deterministic vector function and h is a $k \times k$ matrix process. Such models admit a Markov representation of the dynamics of the yield curve involving $k + \frac{1}{2}k(k+1)$ state variables. This simplifies very much their simulation, which can be done either using Monte Carlo or finite difference methods [4,8].

We consider in this note the one-factor quasi-Gaussian HJM model with volatility specification $\sigma_f(t,T) = k(t,T)\sigma(r_t)$ where $k(t,T) = e^{-\beta(T-t)}$, and $\sigma(r_t)$ is the volatility of the short rate $r_t = f(t,t)$. This model admits a two state Markov representation.

It has been noted in [12,10] that in HJM models with log-normal volatility specification, that is for which $\sigma_f(t,T) = \sigma(t,T)f(t,T)$, the rates explode to infinity with probability one, and zero coupon bond prices are zero. It is natural to ask if a similar explosive phenomenon is present also in the quasi-Gaussian HJM model with log-normal volatility $\sigma(r_t) = \sigma r_t$. This model is used in financial practice for modeling swaption volatility smiles [6] and is a particular case of a more general parametric representation [7].

We study in this note the qualitative behavior of the solutions of this model. In the small-noise deterministic limit, we show rigorously that the short rate may explode to infinity in a finite time. More precisely, for sufficiently small mean-reversion β, the deterministic approximation for the short rate has an explosion in finite time, and an upper bound is given on the explosion time, which is saturated in the flat forward rate limit. When Brownian noise is taken into account, the explosion time has a distribution around the deterministic limit.

This phenomenon has implications for the practical use of the model for pricing and simulation. It implies an explosion of the Eurodollar futures prices in this model, and introduces a limitation in the applicability of the model for pricing these products to maturities smaller than the explosion time.

2. Log-normal quasi-Gaussian HJM model

The one-factor log-normal quasi-Gaussian HJM model is defined by the volatility specification

$$\sigma_f(t,T) = \sigma r_t e^{-\beta(T-t)}. \qquad (2)$$

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The simulation of the model requires the solution of the stochastic differential equation for the two variables $\{x_t, y_t\}_{t\geq 0}$ [14,1]

$$dx_t = (y_t - \beta x_t)dt + \sigma(\lambda(t) + x_t)dW_t,$$
$$dy_t = (\sigma^2(\lambda(t) + x_t)^2 - 2\beta y_t)dt, \tag{3}$$

with initial condition $x_0 = y_0 = 0$. Here $\lambda(t) = f(0, t)$ is the forward short rate, giving the initial yield curve. The zero coupon bonds are

$$P(t, T) = \frac{P(0, T)}{P(0, t)} \exp\left(-G(t, T)x_t - \frac{1}{2}G^2(t, T)y_t\right), \tag{4}$$

with $G(t, T) \geq 0$ a non-negative deterministic function [1]. The short rate is $r_t := f(t, t) = \lambda(t) + x_t$. Eqs. (3) can be expressed in terms of the short rate as

$$dr_t = (y_t - \beta r_t + \beta\lambda(t) + \lambda'(t))dt + \sigma r_t dW_t,$$
$$dy_t = (\sigma^2 r_t^2 - 2\beta y_t)dt, \tag{5}$$

with the initial condition $r_0 = \lambda_0 := \lambda(0) > 0$ and $y_0 = 0$.

The solutions of the process (5) may explode with non-zero probability. This will be discussed in a future paper [13]. When the volatility $\sigma = 0$, there is no explosion. Indeed, when $\sigma = 0$, we have

$$dr_t = (y_t - \beta r_t + \beta\lambda(t) + \lambda'(t))dt,$$
$$dy_t = -2\beta y_t dt,$$

with the initial condition $r_0 = \lambda_0$ and $y_0 = 0$. Thus $y_t \equiv 0$, which gives $r_t' = -\beta r_t + \beta\lambda(t) + \lambda'(t)$. This ODE can be easily solved with the result $r_t = \lambda(t)$.

3. Deterministic approximation

Instead of studying directly the distribution of the explosion time of the process (r_t, y_t), we study a deterministic proxy of Eqs. (5). In the limit when the Brownian noise in these equations goes to zero, then $(r_t, y_t) \to (r(t), y(t))$, where $(r(t), y(t))$ satisfy the two-dimensional ODE:

$$r'(t) = y(t) - \beta r(t) + \beta\lambda(t) + \lambda'(t),$$
$$y'(t) = \sigma^2 r^2(t) - 2\beta y(t), \tag{6}$$

with $r(0) = \lambda_0$ and $y(0) = 0$. The variable $r(t)$ can be interpreted as the deterministic approximation of the short rate r_t and its expected value $\mathbb{E}^{\mathbb{Q}}[r_t]$ for the small-noise limit. The pair $(r(t), y(t))$ is a deterministic approximation of the two-dimensional SDE (5).

We study here the qualitative properties of the solution for $r(t)$. Even though (6) is a system of 2D ODEs, we will show that $r(t)$ can be expressed as a solution to a 1D integral equation.

Proposition 1. *$r(t)$ satisfies the integral equation*

$$r(t) = \lambda(t) + \frac{\sigma^2}{\beta}\int_0^t r^2(s)[e^{\beta(s-t)} - e^{2\beta(s-t)}]ds. \tag{7}$$

Proof. We can solve for $y(t)$ as

$$y(t) = \sigma^2 \int_0^t r^2(s)e^{2\beta(s-t)}ds. \tag{8}$$

Substituting into (6) we get

$$r'(t) + \beta r(t) = \sigma^2 \int_0^t r^2(s)e^{2\beta(s-t)}ds + \beta\lambda(t) + \lambda'(t). \tag{9}$$

Multiplying by the integrating factor $e^{\beta t}$ and integrating from 0 to t, we obtain:

$$\begin{aligned} r(t)e^{\beta t} - \lambda_0 &= \sigma^2 \int_0^t \int_0^u r^2(s)e^{2\beta s}e^{-\beta u}duds + \lambda(t)e^{\beta t} - \lambda_0 \\ &= \lambda(t) + \sigma^2 \int_0^t \int_s^t r^2(s)e^{2\beta s}e^{-\beta u}duds - \lambda_0, \end{aligned} \tag{10}$$

which yields Eq. (7). □

We show next that if $\lambda(t)$ is uniformly bounded, for sufficiently large β or sufficiently small σ, $r(t)$ is also uniformly bounded, and hence there will be no explosion.

Proposition 2. *Assume that $\lambda(t)$ is uniformly bounded. Then, for sufficiently large β or sufficiently small σ, we have*

$$\max_{t\geq 0} r(t) \leq \frac{\beta^2}{\sigma^2}\left(1 - \sqrt{1 - \max_{t\geq 0}\lambda(t)\frac{2\sigma^2}{\beta^2}}\right). \tag{11}$$

It follows that there will be no explosion.

Proof. We only give a proof for the large β result. The same result holds for sufficiently small σ with a very similar proof.

For any $t \in [0, T]$, we have from Eq. (7)

$$r(t) \leq \max_{0\leq t\leq T}\lambda(t) + \left[\max_{0\leq t\leq T} r(t)\right]^2 \frac{\sigma^2}{\beta}\int_0^\infty [e^{-\beta s} - e^{-2\beta s}]ds, \tag{12}$$

which implies that $R(T) := \max_{0\leq t\leq T} r(t)$ satisfies

$$R(T) - R^2(T)\frac{\sigma^2}{2\beta^2} \leq \max_{0\leq t\leq T}\lambda(t). \tag{13}$$

This implies that we have either (i) $R(T) \leq R_1(T)$, or (ii) $R(T) \geq R_2(T)$, with

$$R_{1,2}(T) := \frac{\beta^2}{\sigma^2}\left(1 \mp \sqrt{1 - \max_{0\leq t\leq T}\lambda(t)\frac{2\sigma^2}{\beta^2}}\right). \tag{14}$$

For large β, $R(T)$ is bounded by Proposition 2, while $R_2(T) \to \infty$ as $\beta \to \infty$. Therefore, for sufficiently large $\beta > 0$, we have $R(T) \leq R_1(T)$. Taking now $T \to \infty$, we have by the uniformly bounded assumption $\max_{t\geq 0}\lambda(t) < \infty$. It follows that for sufficiently large β,

$$\max_{t\geq 0} r(t) \leq \frac{\beta^2}{\sigma^2}\left(1 - \sqrt{1 - \max_{t\geq 0}\lambda(t)\frac{2\sigma^2}{\beta^2}}\right). \tag{15}$$

We conclude that for sufficiently large β, $r(t)$ is not explosive and is indeed uniformly bounded as long as $\lambda(t)$ is uniformly bounded. □

Remark 1. From Proposition 2, it follows that $\max_{t\geq 0} r(t)$ is uniformly bounded as either $\beta \to \infty$ or $\sigma \to 0$, since

$$\begin{aligned} \limsup_{\beta\to\infty}\max_{t\geq 0} r(t) &\leq \limsup_{\beta\to\infty}\frac{\beta^2}{\sigma^2}\left(1 - \sqrt{1 - \max_{t\geq 0}\lambda(t)\frac{2\sigma^2}{\beta^2}}\right) \\ &= \max_{t\geq 0}\lambda(t), \end{aligned}$$

and the same result holds for $\sigma \to 0$.

From Proposition 1, Proposition 2 and Remark 1, we immediately get the following corollary.