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On the probability of union in the n-space

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ABSTRACT

In this paper, our sets are orthants in R^n and N, the number of them, is large ($N > n$). We introduce the modified inclusion–exclusion formula in order to efficiently calculate the probability of a union of such events. The new formula works in the bivariate case, and can also be used in R^n, $n \geq 3$ with a condition on the projected sets onto lower dimensional spaces. Numerical examples are presented.

1. Introduction

Let $A_1, \ldots, A_N$ be events in an arbitrary probability space. In many applications we are interested to find probabilities of Boolean functions of them. Boolean functions of events are used in reliability theory, where consecutive events play an important role. The classical results to obtain probabilities of Boolean functions of $A_1, \ldots, A_N$ are the following. Inclusion–exclusion formula:

$$P(A_1 \cup \cdots \cup A_N) = S_1 - S_2 + \cdots + (-1)^{N-1} S_N, \tag{1}$$

where

$$S_k = \sum_{1 \leq i_1 < \cdots < i_j \leq k} P(A_{i_1} \ldots A_{i_j}), \quad k = 1, \ldots, N$$

and the formulas for $P_{(r)}$, the probability that at least r occur and $P_{[r]}$, the probability that exactly r occur. We have:

$$P_{(r)} = \sum_{i=r}^{N} (-1)^{i-r} \binom{r-1}{i-1} S_i, \qquad P_{[r]} = \sum_{i=r}^{N} (-1)^{i-r} \binom{r}{i} S_i. \tag{2}$$

In the literature (1) is frequently attributed to [8,5]. However, [4] already obtained (1) and (2) as a special case.

Bounds for the probability of the union were given by [2]: $P\left(\sum_{i=1}^{N} A_i\right) \leq S_1$. [1] generalized it and also gave lower bounds for the same probability. His bounds are:

$$P\left(\sum_{i=1}^{N} A_i\right) \leq S_1 - S_2 + \cdots + (-1)^r S_r, \quad \text{if } r \text{ is odd}$$

$$P\left(\sum_{i=1}^{N} A_i\right) \geq S_1 - S_2 + \cdots + (-1)^r S_r, \quad \text{if } r \text{ is even.}$$

[3] observed that if we use only S_1, S_2, then sharp bounds for the probability of the union can be given:

$$P\left(\sum_{i=1}^{N} A_i\right) \geq \frac{2}{j+1} S_1 - \frac{2}{j(j+1)} S_2, \quad \text{where } j = 1 + \left\lfloor \frac{2S_2}{S_1} \right\rfloor. \tag{3}$$

[6,7] used linear programming to prove (3) and also gave sharp lower and upper bounds for the union, using S_1, S_2, S_3. [9,11,12] observed that the sharp probability bounds, using $S_1, \ldots, S_m$, are optimum values of LP's that he called binomial moment problems. In doing so, he opened a new research area: the discrete moment problems. The term binomial moment comes from the fact that if ξ is the number of events in $A_1, \ldots, A_N$, which occur, then

$$S_k = E\left[\binom{\xi}{k}\right], \quad k = 1, \ldots, N. \tag{4}$$

By convention we write $S_0 = 1$ with which (4) holds also for $k = 0$. Eq. (4) is a classical theorem and it is not known who proved it first.

Our events in this paper are orthants in the n-space, designated by

$$A(z^{(1)}), \ldots, A(z^{(N)}), \tag{5}$$

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where $z^{(1)}, \ldots, z^{(N)}$ are the vertices of the orthants ($z^{(i)} \in R^n$, $i = 1, \ldots, N$). We assume that $z^{(1)}, \ldots, z^{(N)}$ is an antichain in the partially ordered set R^n, i.e., for no i, j ($i \neq j$) do we have $z^{(i)} \leq z^{(j)}$. Important example for such sets are the p-efficient points of a discrete distribution (introduced by [10]). If $\xi = (\xi_1, \ldots, \xi_n)$ and the support set of ξ_i is $Z_i = (z_{i0}, \ldots, z_{ik_i})$, $i = 1, \ldots, n$, then we create the Cartesian product $Z = Z_1 \times \cdots \times Z_n$. Union of such events can be written as:

$$\bigcup_{i=1}^{N} A_i, \quad \text{where } A_i = A(z^{(i)}) = \{x \in R^n \mid x \leq z^{(i)}\}, \quad i = 1, \ldots, N, \tag{6}$$

where $z^{(i)}$'s are the vertices of the orthants in R^n.

We derive the new formulas for the probability of (6) starting from the bivariate case.

2. The bivariate case

For the sake of completeness we list some basic definitions in connection with partially ordered sets (or *poset*, for short). Let two elements x and y be in a poset. We say that x and y are *comparable* if $x \leq y$ or $y \leq x$. Otherwise x and y are *incomparable*. An element M is *maximal* if $M \leq x \rightarrow M = x$ and an element m is *minimal* if $x \leq m \rightarrow m = x$. We say that y *covers* x, denoted $x \lessdot y$, if $x < y$ and no element between them.

Definition 1. A rank function $r(\cdot)$ of a poset P is function $r : P \rightarrow \{0\} \cup \mathbb{N}$ having the following properties:

(i) if s is minimal, then $r(s) = 0$.
(ii) if t covers s (i.e., $t \gtrdot s$), then $r(t) = r(s) + 1$.

Now, together with the presented basic terms regarding posets, allow us to introduce a counting measure—*reverse rank function* ρ, as follows.

Definition 2. On a finite poset P, with n maximal elements the reverse rank function $\rho : P \rightarrow \{1, \ldots, n\}$ is defined by:

$$\rho(E) = \sum_{i=1}^{n} \mathbb{1}_{E \subseteq M_i},$$

where E is any element of P and M_i's are the incomparable maximal elements of P.

Note that the reverse rank function $\rho(E)$ can be used as a counting measure, and it returns the number of maximal elements containing E.

Example 1. Consider the following incomparable sets: $A_1 = A((2, 10)), A_2 = A((4, 7)), A_3 = A((7, 5)), A_4 = A((11, 1))$. Their intersections and the corresponding reverse rank function values are as follows:

$$\begin{array}{ll}
A_1A_2 = \{x \mid x \leq (2, 7)\} & \rho(A_1A_2) = 2 \\
A_1A_3 = \{x \mid x \leq (2, 5)\} & \rho(A_1A_3) = 3 \\
A_1A_4 = \{x \mid x \leq (2, 1)\} & \rho(A_1A_4) = 4 \\
A_2A_3 = \{x \mid x \leq (4, 5)\} & \rho(A_2A_3) = 2 \\
A_2A_4 = \{x \mid x \leq (4, 1)\} & \rho(A_2A_4) = 3 \\
A_3A_4 = \{x \mid x \leq (7, 1)\} & \rho(A_3A_4) = 2 \\
A_1A_2A_3 = \{x \mid x \leq (2, 5)\} & \rho(A_1A_2A_3) = 3 \\
A_1A_2A_4 = \{x \mid x \leq (2, 1)\} & \rho(A_1A_2A_4) = 4 \\
A_1A_3A_4 = \{x \mid x \leq (2, 1)\} & \rho(A_1A_3A_4) = 4 \\
A_2A_3A_4 = \{x \mid x \leq (4, 1)\} & \rho(A_2A_3A_4) = 3 \\
A_1A_2A_3A_4 = \{x \mid x \leq (2, 1)\} & \rho(A_1A_2A_3A_4) = 4.
\end{array} \tag{7}$$

Refer to Fig. 1 for two different ways of ordering: (a) by subset relation and (b) by the reverse rank function and the inclusion. On a poset with length $n - 1$ in R^2 (i.e., every maximal chain has the same length of $n - 1$) ordered by inclusion, we have the following relations.

(8)

rank $r(\cdot)$	reverse rank $\rho(\cdot)$	group of elements
$n-1$	1	maximal elements
$n-2$	2	incomparable pairs
$n-3$	3	incomparable triplets and some pairs
$\vdots$	$\vdots$	$\vdots$
0	n	minimal element and others.

From relation (8), if we remove all the current maximal elements, then the elements in the reverse rank of 2 in (8) – i.e., incomparable pairwise intersections – will be the new maximal elements. The new poset can be written as the following.

(9)

rank $r(\cdot)$	reverse rank $\rho'(\cdot)$	group of elements
		maximal elements (removed)
$n-2$	1	incomparable pairs (new maximal)
$n-3$	2	incomparable triplets and some pairs
$\vdots$	$\vdots$	$\vdots$
0	$n-1$	the minimal element and all others.

Subtracting component-wise (9) from (8), the reverse rank becomes 1 (the second column values) for every anti-chain. Equivalently, we have $\rho - \rho' = 1$ for all elements of the poset. This means that every element in the entire poset is counted exactly once by the operation. Incomparable pairwise intersections can easily be found by:

Algorithm to enumerate incomparable pairs

Step 0. Sort n maximal points $(x_1, y_1), \ldots, (x_n, y_n)$ on their 1st component such that $x_1 < \cdots < x_n$. Then we have the relation between components:

$$\begin{array}{l} x_1 < \cdots < x_n \\ y_1 > \cdots > y_n. \end{array} \tag{10}$$

Step 1. Then the $n - 1$ "incomparable" pairs are:

$$Z_i = \{z \in R^2 \mid z \leq (x_i, y_{i+1})\}, \quad i = 1, \ldots, n - 1,$$

which can be written up in the following form:

$$Z_1 = A((x_1, y_2)), Z_2 = A((x_2, y_3)), \ldots, Z_{n-1} = A((x_{n-1}, y_n)). \tag{11}$$

We are ready to introduce the following:

Theorem 1 (*Modified Inclusion–Exclusion Formula for the Bivariate Case*)**.** *In any given probability space, on a finite poset in R^2 with the maximal elements $A_i = A(s^{(i)}) = \{z \in R^2 \mid z \leq s^{(i)}\}$, $i = 1, \ldots, N$, sorted as in (10), we have the formula:*

$$P\left(\bigcup_{i=1}^{N} A_i\right) = \sum_{i=1}^{N} P(A_i) - \sum_{i=1}^{N-1} P(A_iA_{i+1}) = S_1 - S_2', \tag{12}$$

where S_1 is the first binomial moment of the events $A_1, \ldots, A_N$, and S_2' is the sum of the probabilities of the "incomparable" pairwise intersections.

Proof.

$$P\left(\bigcup_{i=1}^{N} A_i\right) = P(A_1) + P(A_2\bar{A}_1) + P(A_3\bar{A}_1\bar{A}_2) + \cdots + \cdots + P(A_k\bar{A}_1\bar{A}_2 \ldots \overline{A_{k-1}}) + \cdots + P(A_N\bar{A}_1\bar{A}_2 \ldots \overline{A_{N-1}})$$