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Protection of flows under targeted attacks

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ABSTRACT

We present a new robust optimization model for the problem of maximizing the amount of flow surviving the attack of an interdictor. Given some path flow, our model allows the interdictor to specify the amount of flow removed from each path individually. In contrast to previous models, for which no efficient algorithms are known, the most important basic variants of our model can be solved in poly-time. We also consider extensions where there is a budget to set the interdiction costs.

1. Introduction

Network flow problems form one of the most important classes of optimization problems with numerous real-world applications, e.g., in production systems, logistics, and communication networks. The increasing dependence of our society on constant availability of such network services motivates the study of new flow models that are *robust* against unforeseen interferences, link failures, and targeted attacks by external forces.

The theory of *robust optimization* offers various techniques to handle the issue of planning in face of uncertainties and unreliability; see, e.g., [3,4] for surveys. A general idea is to model uncertainty by a set of possible scenarios Ω that is specified along with the instance of the optimization problem under consideration, where each scenario represents a possible outcome involving failures in the infrastructure, intentional sabotage, or similar complications. With respect to a worst-case analysis, the *robust* objective value of a feasible solution x is the worst objective value of x among all possible scenarios $z \in \Omega$.

Robust optimization can thus be interpreted as a two-player game: the first player ("the decision-maker") chooses a solution x from a set X of feasible solutions to the underlying "nominal" optimization problem. Afterwards, the second player ("the adversary" or "interdictor") selects a scenario z from the predefined scenario set Ω. While the first player aims at maximizing the resulting objective value $\mathrm{val}(x, z)$, the adversary selects $z \in \Omega$ in order to reduce $\mathrm{val}(x, z)$ as far as possible. The robust optimization problem therefore asks for an optimal $x \in X$ solving

$$\max_{x\in X} \min_{z\in\Omega} \mathrm{val}(x, z).$$

In this paper, we present a new robust optimization model for network flows. In existing models the interdictor acts on a subset of the arcs of the network and the interdiction of an arc affects all flow on that arc equally. By contrast, our model allows the interdictor to specify the amount of flow removed from each flow path individually (we therefore deal with flows on *paths*). In this context, it might be helpful to think of the interdictor as a thief who steals particular flow units of his choice: As an illustrative example, consider a train network in which each flow path represents a train and train robbers try to remove as much cargo as possible from the trains, attacking each train at the most vulnerable station it traverses (possibly sparing other trains that go through the same station). Besides providing this new perspective

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of robust flow optimization, the new model has the advantage that optimal robust flows can be computed in polynomial time. We also consider further variants of the problem, in which the flow player can adjust the protection for the flow he sends through the network on each arc, subject to a budget. While we focus our discussion throughout this paper mainly on the classic maximum flow problem, we also point out that our results directly extend to robust optimization versions of a general class of packing problems that extends beyond network flows.

Contribution and structure of the paper. In the remainder of this section, we present our new robust flow model and compare it to existing models. We also discuss literature on the closely related field of network interdiction.

In Section 2, we study the basic version of our new robust flow model, in which interdiction costs for the arcs of the network are given and the flow player determines a path flow with the goal of maximizing the surviving flow value after interdiction. We show that the optimal strategy for the flow player can be found by solving a parametric LP, where the parameter corresponds to the cost of the most expensive arc affected by the interdictor. We also show that in general, optimal solutions to our problem are not integral, and that any combinatorial algorithm for our problem can also be used to solve a feasibility version of the (fractional) multicommodity flow problem combinatorially. Finally, we point out that our results still hold when the flow player's options are limited by a budget, and further extend to a very general class of packing problems, including multicommodity flows, abstract flows, and b-matchings.

In Section 3, we study a design variant of the problem, where the flow player has to buy the protection of the flow he sends through the network subject to a limited budget. For each arc, the cost of protection is proportional to the chosen interdiction cost on that arc and the amount of flow that needs to be protected. We show that this seemingly hard non-linear optimization problem can be solved by exploiting insights on the structure of an optimal solution.

In Section 4, we discuss a generalization of the problem from the preceding section, in which an initial (free) protection of the flow on each arc is given but the interdiction costs can be further increased by the flow player subject to his budget. We show, that in contrast to the problems discussed earlier, this problem is not only *NP*-hard but does not even allow for approximation algorithms.

1.1. The new model

We are given a directed graph $D = (V, A)$ with source $s \in V$ and sink $t \in V$, and arc capacities $u \in \mathbb{Z}_+^A$. We consider flows on paths, as in [6, Section 4]. Let $\mathcal{P}$ denote the collection of all s–t-paths in D. The strategy choices of the decision maker (whom we call the *flow player*) are given by the set

$$X := \left\{x \in \mathbb{R}_+^{\mathcal{P}} \mid \sum_{P \in \mathcal{P}: e \in P} x_P \le u_e \ \forall e \in A\right\}$$

of all feasible s–t-flows in the capacitated network (D, u), i.e., the flow player specifies the amount of flow along each s–t-path subject to the arc capacities.

Classic robust flow models (which are discussed further below) are built on the assumption that the interdictor attacks arcs of the network subject to a budget, equally affecting all flow paths traversing the interdicted arcs. In contrast, in our model we instead think of the interdictor as a thief who might directly attack and steal flow on each individual path rather than manipulating an arc e as a whole. Each arc $e \in A$ is equipped with an *interdiction cost* $c_e \ge 0$, specifying the cost of stealing one unit of flow on that arc. The interdictor can, after the flow player has chosen a flow $x \in X$, use a given budget B_I in order to steal flow on some of the paths. Therefore the interdictor chooses a scenario/strategy

$$z \in \Omega := \left\{z \in \mathbb{R}_+^{A \times \mathcal{P}} \mid \sum_{e \in A} c_e \sum_{P \in \mathcal{P}: e \in P} z_{e,P} \le B_I\right\}.$$

The remaining flow after applying the interdiction strategy z to flow x is defined by $\bar{x}_P := (x_P - \sum_{e \in P} z_{e,P})^+$ for each $P \in \mathcal{P}$. The flow player's goal is to maximize $\operatorname{val}(x, z) := \sum_{P \in \mathcal{P}} \bar{x}_P$, anticipating the interdictor's response, who wants to minimize the same quantity, i.e., steal as much flow as possible.

Note that an attack on a particular path $P \in \mathcal{P}$ should always happen on a cheapest arc $e \in P$. Therefore, after the flow $x \in X$ has been chosen, an optimal strategy for the interdictor is the following greedy approach: Sort the paths $P \in \mathcal{P}$ in order of non-decreasing bottleneck cost $\bar{c}_P := \min_{e \in P} c_e$ and steal flow along the paths in this order until the budget B_I has been used up.

This tractability of the interdictor's optimal strategy is a desirable property of our model as it allows computation of the robust value of any given flow. Note that, in contrast, for the models in [6,19] discussed below, the interdictor's optimal answer to a given flow is *NP*-hard to compute (in both cases, the interdictor's problem is equivalent to the budgeted maximum coverage problem [6]).

Also, with the exception of the basic model in [2], no efficient algorithm or constant factor approximation is known for the flow player's problem in the robust flow models discussed below despite intense research. We will show that for our model, both the maximum flow version as well as the design version (in which the flow player adjusts the protection of the links in the network) can be solved efficiently. Furthermore, our model and these positive algorithmic results naturally extend to a very general class of packing linear programs, including, e.g., multicommodity flows, abstract flows, and b-matchings, and allows the easy integration of additional budget constraints.

1.2. Related work

In the following, we discuss existing robust flow models and the related concept of network interdiction.

Robust flows. Robust flows subject to cost uncertainties were studied by Bertsimas and Sim [7]. Aneja et al. [2] started the study of robust maximum (path) flows in presence of an interdictor (capacity uncertainty), who in their model could remove a single arc from the network. The goal of the flow player, as in all subsequent papers, is to maximize the value of the surviving flow. Aneja et al. showed that the problem can be solved in polynomial time using a parametric LP. Du and Chandrasekaran [12] showed that, as soon as the interdictor is allowed to remove two arcs, the corresponding dual separation problem becomes *NP*-hard: However, as pointed out in [11], this does not imply that the robust flow problem itself is *NP*-hard. While the complexity of the problem with two arcs is open, it is shown in [11] that the problem is *NP*-hard when the number of arcs that the interdictor is allowed to remove is not bounded by a constant. On the positive side, Bertsimas et al. [6] building upon a generalization of the parametric LP used in [2], gave an LP-based approximation, in terms of the amount of flow removed by the interdictor in an optimal solution, for the case where the interdictor can remove any given number of arcs B_I. More recently, Bertsimas et al. [5] showed that the same flow also yields a $1 + (B_I/2)^2/(B_I + 1)$-approximation. Formally, the model in [6,2,12,5] is defined for the uncertainty set $\Omega = \{z \in \{0, 1\}^A \mid 1^T z \le B_I\}$ and the problem that has to be solved by the flow player is $\max_{x \in X} \min_{z \in \Omega} \operatorname{val}(x, z)$, where $\operatorname{val}(x, z)$ is the amount of flow $x \in X$ that survives after the interdictor selects the scenario $z \in \Omega$, i.e., $\operatorname{val}(x, z) = \sum_{P \in \mathcal{P}}(1 - \max_{e \in P} z_e)x_P$. A fractional version of this uncertainty set was proposed in [8]. Each