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A coradiant based scalarization to characterize approximate solutions of vector optimization problems with variable ordering structures

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ABSTRACT

This paper investigates some properties of approximate efficiency in variable ordering structures where the variable ordering structure is given by a special set valued map. We characterize ε-minimal and ε- nondominated elements as approximate solutions of a multiobjective optimization problem with a variable ordering structure and give necessary and sufficient conditions for these solutions, via scalarization.

1. Introduction

One of major problems of the optimization is that a decision maker should simultaneously obtain the optimal solution of several objectives which are in conflict. Such problems can be formulated in the form of multiobjective optimization problems. Applications of multiobjective optimization problems can be found in financial mathematics, economic theory, management science, design engineering and many other fields. In multiobjective optimization, it is usual to define the partial ordering by a fixed cone in the objective space. But sometimes decisions may depend on more that one objective and what is preferred may vary on the actual state. Already in 1974, Yu introduced the concept of variable ordering structure by using several different cones in vector optimization [25], and defined optimal solutions called nondominated solutions. In the literature, variable ordering structures usually were introduced by cone valued maps. These maps associate a cone to any objective vector. The variable ordering structure leads to various concepts of optimality in multiobjective optimization. In [3,7,9,8,25], different concepts of optimality and their properties have been studied. Some applications of the multiobjective optimization problem with variable ordering structures can be found in [23,24].

In most cases, finding an exact optimal solution of a multiobjective optimization problem may be very hard. Different numerical algorithms are often applied to solve a given optimization problem. Numerical algorithms as usual provide approximate optimal solution. Hence, the study of approximate solutions is of interest. One of the first concepts of approximate solutions for multiobjective optimization problems is introduced by Kutateladze [19]. Later, different concepts of approximate solutions were introduced (see for instance [2,4,11,20]). Gutierrez et al. using the concept of the coradiant set, introduced a new concept of approximate solutions and showed that many previous concepts of approximate solutions are special cases of this new definition [12,13]. In [1,21], authors have generalized the concept of ε-efficiency (in the sense of Kutateladze) to a similar concept in variable ordering structures. They defined a set valued map $C : Y \rightrightarrows Y$ such that $C(y)$ is a closed set and $0 \in \partial(C(y))$. Using this map, they proposed the concept of ϵk^0-nondominated points and ϵk^0-minimizers in vector optimization problem with variable ordering structures.

In this work, we study approximate solutions of multiobjective problems by using a variable ordering structure which associates a coradiant set to any element of the objective space. By using the augmented dual cones given by Kasimbeyli in [15], we introduce the concepts of ε-dual coradiant set and augmented ε-dual coradiant set in this work. The elements of these sets are used to construct scalarizing functions which can be viewed as generalizations of the monotone sublinear functions introduced in [15,16] (see also [10]). Kasimbeyli has introduced the conic

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scalarization method and characterized efficient solutions of vector optimization problems without convexity and boundedness conditions [15,16]. In this work, we introduce the scalarization approach to characterize approximate efficient elements of multiobjective optimization problems with variable ordering structure.

The rest of the paper is organized as follows. Section 2 gives some notions and preliminaries. In Section 3, we present a special class of the variable ordering structures and introduce new concepts of approximate solutions for multiobjective optimization problem with variable ordering structure. Section 4 presents a scalarization method to characterize approximate solutions in variable ordering structures.

2. Preliminaries

Throughout the paper, $int(G)$, $cl(G)$, $bd(G)$ and $co(G)$ denote the interior, the closure, the boundary and the convex hull of a set $G \subseteq \mathbb{R}^p$, respectively. A set $G \subseteq \mathbb{R}^p$ is called cone if $\alpha G = G$ for all $\alpha > 0$. The set $cone(G) = \{\alpha g \mid g \in G, \alpha > 0\}$ denotes the cone generated by a set G. We say that a cone G is convex if $G + G \subseteq G$, G is solid if $\mathrm{int}(G) \neq \emptyset$ and G is pointed if $G \cap (-G) \subseteq \{0\}$, i.e. if $G \cap (-G) = \{0\}$ when $0 \in G$ and if $G \cap (-G) = \emptyset$ when $0 \notin G$.

Let D be a convex, closed and pointed cone of $\mathbb{R}^p$, which defines a partial ordering on $\mathbb{R}^p$ in the following form:

$$y_1 \leq y_2 \Leftrightarrow y_2 - y_1 \in D.$$

In this paper the following multiobjective optimization problem is considered:

$$\min \{f(x) \mid x \in S\}, \tag{1}$$

where $f : S \to \mathbb{R}^p$ and $S \subseteq \mathbb{R}^n$. Let $A := f(S)$, we recall that $\bar{y} \in A$ is an efficient element of (1) with respect to (w.r.t.) the cone D if $(\bar{y} - D) \cap A \subseteq \{\bar{y}\}$.

In recent decades, many researchers investigated approximate solutions of problem (1). Various concepts of approximate solutions have been introduced. A new notion for the approximate solutions, by using the concept of coradiant sets (instead of a cone in the classical definition) was given in [13]. Now we recall the definition of the coradiant set.

Definition 2.1 (*[13]*). The set $C \subseteq \mathbb{R}^p$ is called a coradiant set if $\alpha C \subseteq C$ for all $\alpha \geq 1$.

Throughout this paper, we assume that $0 \notin C$ and the coradiant set C is pointed if $C \cap (-C) = \emptyset$. For given $\varepsilon > 0$, let $C(\varepsilon) := \{\varepsilon c \mid c \in C\}$.

Definition 2.2 (*[13]*). Let $\varepsilon > 0$. $\bar{y} \in A$ is an ε-efficient element of A w.r.t. coradiant set C if

$$(\bar{y} - C(\varepsilon)) \cap A \setminus \{\bar{y}\} = \emptyset.$$

Let C be a coradiant set, and let $\varepsilon > 0$ be a given positive number. The sets

$$C^*(\varepsilon) := \left\{\ell \in \mathbb{R}^p \mid \langle \ell, y\rangle \geq 0 \text{ for all } y \in C(\varepsilon)\right\},$$
$$C^\#(\varepsilon) := \left\{\ell \in \mathbb{R}^p \mid \langle \ell, y\rangle > 0 \text{ for all } y \in C(\varepsilon)\right\}$$

are called the ε-dual coradiant set of C and its quasi-interior, respectively. Let $\lambda > 0$ be a given positive number. For the coradiant set C, the augmented ε-dual coradiant set and its quasi-interior are denoted by $C^{\lambda *}(\varepsilon)$ and $C^{\lambda \#}(\varepsilon)$, respectively, and are defined as follows:

$$C^{\lambda *}(\varepsilon) := \left\{(\ell, \alpha) \in C^\#(\varepsilon) \times \mathbb{R}_+ \mid \langle \ell, y\rangle - \alpha \|y\| \geq \lambda \text{ for all } y \in C(\varepsilon)\right\}.$$
$$C^{\lambda \#}(\varepsilon) := \left\{(\ell, \alpha) \in C^\#(\varepsilon) \times \mathbb{R}_+ \mid \langle \ell, y\rangle - \alpha \|y\| > \lambda \text{ for all } y \in C(\varepsilon)\right\}.$$

3. Variable ordering structure

Variable ordering structure in vector optimization was firstly introduced by Yu in [25]. He suggested a variable cone instead of a fixed one in the evaluation process. After Yu, the multiobjective optimization problem with variable ordering structures was studied intensively. Motivated by these considerations, different concepts of optimality have been introduced in multiobjective optimization with variable ordering structure.

In this section, we study approximate efficiency in multiobjective optimization with variable ordering structure on $\mathbb{R}^p$, which is defined by a set valued map $\mathcal{C} : A \to 2^A$ with $\mathcal{C}(y) \subseteq \mathbb{R}^p$ is a closed, convex and pointed coradiant set such that $0 \notin \mathcal{C}(y)$ for every $y \in A$. For convenience, in the sequel we will use $C_y := \mathcal{C}(y)$. Let ε and λ be positive numbers. Then, the augmented ε-dual coradiant sets for a variable coradiant set C_y, can be defined in a similar way, which was used for a fixed coradiant set, as follows:

$$C_y^{\lambda *}(\varepsilon) := \left\{(\ell, \alpha) \in C_y^\#(\varepsilon) \times \mathbb{R}_+ \mid \langle \ell, z\rangle - \alpha \|z\| \geq \lambda \text{ for all } z \in C_y(\varepsilon)\right\},$$
$$C_y^{\lambda \#}(\varepsilon) := \left\{(\ell, \alpha) \in C_y^\#(\varepsilon) \times \mathbb{R}_+ \mid \langle \ell, z\rangle - \alpha \|z\| > \lambda \text{ for all } z \in C_y(\varepsilon)\right\}.$$

Now we introduce two new concepts of approximate efficient elements in multiobjective optimization with a variable ordering structure.

Definition 3.1. For a given $\varepsilon > 0$, $\bar{y} \in A$ is called an ε-nondominated (weakly ε-nondominated) element of A w.r.t. the map $\mathcal{C}$, if there is no $y \in A$ such that $\bar{y} \in \{y\} + C_y(\varepsilon)$ ($\bar{y} \in \{y\} + \mathrm{int}(C_y(\varepsilon))$).

Definition 3.2. For given $\varepsilon > 0$, $\bar{y} \in A$ is called an ε-minimal (weakly ε-minimal) element of A w.r.t. the map $\mathcal{C}$ if there is no $y \in A$ such that $\bar{y} \in \{y\} + C_{\bar{y}}(\varepsilon)$ ($\bar{y} \in \{y\} + \mathrm{int}(C_{\bar{y}}(\varepsilon))$).

Remark 3.3. From Definition 3.1 it can be concluded that if $\bar{y}$ satisfies the following condition,

$$\bar{y} - C_y(\varepsilon) \cap A = \emptyset \quad \text{for all } y \in A, \tag{2}$$

then $\bar{y}$ is an ε-nondominated element of A w.r.t. the map $\mathcal{C}$. It should be noted that the relation (2) is a sufficient condition for $\bar{y}$ to be an ε-nondominated element.

Remark 3.4. From Definition 3.2 it can be concluded that $\bar{y}$ is an ε-minimal element of A w.r.t. the map $\mathcal{C}$ if and only if

$$\bar{y} - C_{\bar{y}}(\varepsilon) \cap A = \emptyset. \tag{3}$$

The following example demonstrates that, Remarks 3.3 and 3.4 can be used to characterize the ε-minimal and ε-nondominated elements of A w.r.t. coradiant valued map $\mathcal{C}$.

Example 3.5. Let $\varepsilon > 0$, $C_1, C_2 \subseteq \mathbb{R}^2$ be two different closed, convex and pointed coradiant sets and $\mathcal{C} : \mathbb{R}^2 \to \mathbb{R}^2$ be a coradiant valued map as follows:

$$C_y = \begin{cases} C_1 & \text{if } y_1 \geq y_2 \\ C_2 & \text{if } y_1 < y_2. \end{cases}$$

Assume that $\bar{y} \in A$. By Remark 3.4, if $\bar{y} - C_{\bar{y}}(\varepsilon) \cap A = \emptyset$ then $\bar{y}$ is an ε-minimal element of A w.r.t. the coradiant valued map $\mathcal{C}$. In the other word, if

$$\begin{cases} (\bar{y} - C_1(\varepsilon)) \cap A = \emptyset & \text{if } \bar{y}_1 \geq \bar{y}_2, \\ (\bar{y} - C_2(\varepsilon)) \cap A = \emptyset & \text{if } \bar{y}_1 < \bar{y}_2, \end{cases} \tag{4}$$

then $\bar{y} \in A$ is an ε-minimal element of A w.r.t. coradiant valued map $\mathcal{C}$.