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Stochastic comparison in MRL ordering for parallel systems with two exponential components

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ABSTRACT

In this paper, we investigate the stochastic comparison of parallel systems with two independent exponential components in terms of mean-residual (MRL) ordering. We obtain a more general and reasonable sufficient condition for guaranteeing MRL ordering of the systems than the one given in some existing results in the literature.

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1. Introduction

Stochastic comparison of order statistics has many applications and this topic has a long history. See Proschan and Sethuraman [7], Kochar and Rojo [3], Dykstra et al. [2], Kochar and Xu [4], Zhao and Balakrishnan [9], and some recent nice review articles, such as, Balakrishnan and Zhao [1]. As we all know, order statistics is closely related with the lifetimes of k-out-of-n systems. Typically, the largest order statistic corresponds to the lifetime of parallel system.

As we can notice, the exponential models are the most commonly used ones among the lifetime models; the exponential variables are the special cases of some other variables; and, by some increasing transformations, exponential variables can be converted to some other variables, such as, the Weibull variables. For these reasons, in this paper, we confine the investigation in exponential settings.

Let $X_1, \ldots, X_n$ be independent exponential random variables with X_i having hazard rate $\lambda_i, i = 1, \ldots, n$, and $Y_1, \ldots, Y_n$ be another set of independent exponential random variables with Y_i having hazard rate $\mu_i, i = 1, \ldots, n$. Denote $X_{n:n} = \max\{X_1, \ldots, X_n\}$, and $Y_{n:n} = \max\{Y_1, \ldots, Y_n\}$. By symmetry, throughout this paper, we assume $\lambda_1 \leq \cdots \leq \lambda_n$ and $\mu_1 \leq \cdots \leq \mu_n$, and also denote $X_{n:n}$ as $T(\lambda_1, \ldots, \lambda_n)$, $Y_{n:n}$ as $T(\mu_1, \ldots, \mu_n)$ for explicitness.

Denote $\stackrel{m}{\succ}$ as majorization order, $\stackrel{rm}{\succ}$ as reciprocal majorization order, $\geq_{st}$ as the usual stochastic order, and $\geq_{mrl}$ as mean-residual (MRL) order. So far, many results have been established on stochastic orderings between $X_{n:n}$ and $Y_{n:n}$. For example, Pledger and Proschan [6] showed that $(\lambda_1, \ldots, \lambda_n) \stackrel{m}{\succ} (\mu_1, \ldots, \mu_n)$ implies $X_{n:n} \geq_{st} Y_{n:n}$, Dykstra et al. [2] enhanced the above result to reversed hazard rate order, and Misra [5] extended the result to weak majorization order. For $n = 2$, Zhao and Balakrishnan [9] showed that, under the condition $0 < \lambda_1 \leq \mu_1 \leq \mu_2 \leq \lambda_2$,

$$(\lambda_1, \lambda_2) \stackrel{rm}{\succ} (\mu_1, \mu_2) \Longrightarrow X_{2:2} \geq_{mrl} Y_{2:2}. \quad (1)$$

As we know, smaller hazard rate implies longer lifetime, or, better quality of component. By intuition, when $\mu_2 \geq \lambda_2$, the result $X_{2:2} \geq_{mrl} Y_{2:2}$ is more likely to be true. As we can easily confirm, $T(1, 2) \geq_{mrl} T(2, 3)$. We thus believe that the condition $0 < \lambda_1 \leq \mu_1 \leq \mu_2 \leq \lambda_2$ is too stringent and is not so necessary in guaranteeing $X_{2:2} \geq_{mrl} Y_{2:2}$.

As we can easily prove, $T(\lambda, \lambda) \geq_{mrl} (\mu_1, \mu_2)$ when $\mu_2 \geq \mu_1 \geq \lambda$. So, in the sequel, we assume $0 < \lambda_1 < \lambda_2$. For a point (λ_1, λ_2) with $0 < \lambda_1 < \lambda_2$, we define a region $\Gamma_{(\lambda_1,\lambda_2)}$. A point $(x, y) \in \Gamma_{(\lambda_1,\lambda_2)}$, if $0 < x \leq y$, $\frac{1}{y} - \frac{1}{x} \leq \frac{1}{\lambda_2} - \frac{1}{\lambda_1}$, and $\frac{1}{y} + \frac{1}{x} \leq \frac{1}{\lambda_2} + \frac{1}{\lambda_1}$. In other words, $\Gamma_{(\lambda_1,\lambda_2)}$ is the inside part enclosed by the line $y = x$, the curve $\frac{1}{y} - \frac{1}{x} = \frac{1}{\lambda_2} - \frac{1}{\lambda_1}$, and the curve $\frac{1}{y} + \frac{1}{x} = \frac{1}{\lambda_2} + \frac{1}{\lambda_1}$.

Separate the region $\Gamma_{(\lambda_1,\lambda_2)}$ into three parts. Part I is bounded by line $y = \lambda_2$, the line $y = x$, and the curve $\frac{1}{y} + \frac{1}{x} = \frac{1}{\lambda_2} + \frac{1}{\lambda_1}$; part II is bounded by the lines $y - x = \lambda_2 - \lambda_1$, $y = \lambda_2$ and, $y = x$; part III is bounded by the line $y - x = \lambda_2 - \lambda_1$, and the curve $\frac{1}{y} - \frac{1}{x} = \frac{1}{\lambda_2} - \frac{1}{\lambda_1}$. The following picture shows the region $\Gamma_{(\lambda_1,\lambda_2)}$. See Fig. 1.

In this paper, we prove

$$(\mu_1, \mu_2) \in \Gamma_{(\lambda_1,\lambda_2)} \Longrightarrow X_{2:2} \geq_{mrl} Y_{2:2}. \quad (2)$$

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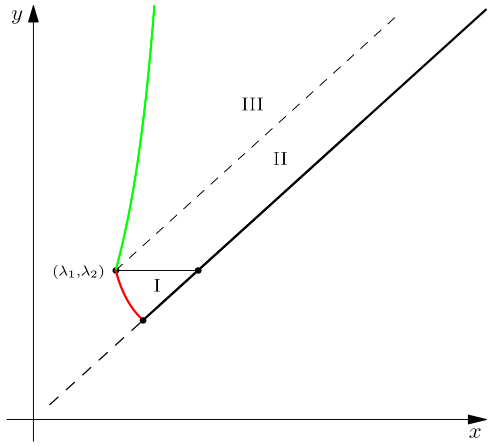

Fig. 1. Region $\Gamma_{(\lambda_1,\lambda_2)}$.

As we can see, the result (1) states that, when (μ_1,μ_2) is in the part I of $\Gamma_{(\lambda_1,\lambda_2)}$, $X_{2:2} \geq_{mrl} Y_{2:2}$. Clearly, the result (2) is an extension of result (1). Since the result (2) does not require $\mu_2 \leq \lambda_2$, we believe the condition $(\mu_1,\mu_2) \in \Gamma_{(\lambda_1,\lambda_2)}$ is a more reasonable sufficient condition for guaranteeing the MRL ordering between $X_{2:2}$ and $Y_{2:2}$. For our convenience, we say (λ_1,λ_2) is g-larger than (μ_1,μ_2) (denote as $(\lambda_1,\lambda_2) \overset{g}{\succ} (\mu_1,\mu_2)$), if $(\mu_1,\mu_2) \in \Gamma_{(\lambda_1,\lambda_2)}$.

The paper is organized as follows. In Section 2, we give the required notations and definitions. Section 3 provides the proof of the result. The proof of a lemma is deferred to Appendix.

2. Notations and definitions

Let X be a nonnegative continuous random variable with distribution function $F_X(t)$, survival function $\bar{F}_X(t) = 1 - F_X(t)$, and density function $f_X(t)$. The reversed hazard function of X is defined as $r_X = f_X/F_X$. For two random variables X and Y, we say X is larger than Y in the usual stochastic order (denoted by $X \geq_{st} Y$), if $\bar{F}_X(t) \geq \bar{F}_Y(t)$; X is larger than Y in reversed hazard rate order (denoted by $X \geq_{rh} Y$), if $r_X(t) \geq r_Y(t)$; and X is larger than Y in MRL order (denoted by $X \geq_{mrl} Y$), if $E(X-t|X>t) \geq E(Y-t|Y>t)$.

Given two vectors $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, b_2, \ldots, b_n)$ with increasing elements, the vector $\boldsymbol{a}$ is said to majorize the vector $\boldsymbol{b}$ (denoted as $\boldsymbol{a} \overset{m}{\succ} \boldsymbol{b}$) if, $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$, and $\sum_{i=1}^k a_i \leq \sum_{i=1}^k b_i$, for $k = 1, \ldots, n-1$. The vector $\boldsymbol{a}$ is said to reciprocal majorize $\boldsymbol{b}$ (denoted as $\boldsymbol{a} \overset{rm}{\succ} \boldsymbol{b}$), if $\sum_{i=1}^j 1/a_i \geq \sum_{i=1}^j 1/b_i$, $j = 1, \ldots, n$. For more details on stochastic orders, see Shaked and Shanthikumar [8].

For our convenience, we denote $A \overset{\text{sgn}}{=} B$ if the signs of A and B are the same. The following lemma will be used in the proof of the main result.

Lemma 2.1. *Let $c(x) = (e^x - 1)/x$. Then, for any $x, y > 0$, $c(y) - (y-x)c(x) > 0$.*

3. Main result

Theorem 3.1. *Given a point (λ_1,λ_2) with $0 < \lambda_1 < \lambda_2$, we have,*

$$(\lambda_1,\lambda_2) \overset{g}{\succ} (\mu_1,\mu_2) \Longrightarrow T(\lambda_1,\lambda_2) \geq_{mrl} T(\mu_1,\mu_2).$$

Proof. Let $X = T(\lambda_1,\lambda_2) = \max\{X_1, X_2\}$, where X_i follows exponential distribution with hazard rate λ_i, $i = 1, 2$, The MRL function of X is

$$\phi(\boldsymbol{\lambda}:t) = \frac{\frac{1}{\lambda_1}e^{-\lambda_1 t} + \frac{1}{\lambda_2}e^{-\lambda_2 t} - \frac{1}{\lambda_1+\lambda_2}e^{-(\lambda_1+\lambda_2)t}}{e^{-\lambda_1 t} + e^{-\lambda_2 t} - e^{-(\lambda_1+\lambda_2)t}} = \frac{\frac{1}{\lambda_1}e^{\lambda_2 t} + \frac{1}{\lambda_2}e^{\lambda_1 t} - \frac{1}{\lambda_1+\lambda_2}}{e^{\lambda_1 t} + e^{\lambda_2 t} - 1}.$$

Consider function

$$\Phi(x_1,x_2) = \frac{\frac{1}{x_1}e^{x_2} + \frac{1}{x_2}e^{x_1} - \frac{1}{x_1+x_2}}{e^{x_1} + e^{x_2} - 1}, \quad 0 < x_1 \leq x_2.$$

To compare $T(\lambda_1,\lambda_2)$ and $T(\mu_1,\mu_2)$ in terms of MRL ordering is equivalent to compare the values of $\Phi(\lambda_1 t, \lambda_2 t)$ and $\Phi(\mu_1 t, \mu_2 t)$ for $t > 0$. Since the direction $(\mu_1 t, \mu_2 t) - (\lambda_1 t, \lambda_2 t)$ is the same as $(\mu_1,\mu_2) - (\lambda_1,\lambda_2)$, so, if Φ is decreasing in the direction $(\mu_1,\mu_2) - (\lambda_1,\lambda_2)$, then, $\Phi(\lambda_1 t, \lambda_2 t) \geq \Phi(\mu_1 t, \mu_2 t)$ holds.

To prove the theorem, we want to show the function $\Phi(x_1,x_2)$ is decreasing along with the vector fields $(1,-1)$, $(1,1)$, $(x_1^2, -x_2^2)$, and (x_1^2, x_2^2).

Denote the numerator part of Φ as N and the denominator part as D. For $i = 1, 2$, by ignoring a common positive factor, we have,

$$\begin{aligned}
\frac{\partial \Phi}{\partial x_1} &= \frac{\partial N}{\partial x_i}D - N\frac{\partial D}{\partial x_i} \\
&= \left[-\frac{1}{x_1^2}e^{x_2} + \frac{1}{x_2}e^{x_1} + \frac{1}{(x_1+x_2)^2}\right](e^{x_1} + e^{x_2} - 1) \\
&\quad - \left[\frac{1}{x_1}e^{x_2} + \frac{1}{x_2}e^{x_1} - \frac{1}{x_1+x_2}\right]e^{x_1} \\
&= -\frac{1}{x_1^2}e^{2x_2} + e^{x_1+x_2}\left(\frac{1}{x_2} - \frac{1}{x_1} - \frac{1}{x_1^2}\right) \\
&\quad + e^{x_2}\left[\frac{1}{x_1^2} + \frac{1}{(x_1+x_2)^2}\right] \\
&\quad + e^{x_1}\left[-\frac{1}{x_2} + \frac{1}{x_1+x_2} + \frac{1}{(x_1+x_2)^2}\right] - \frac{1}{(x_1+x_2)^2},
\end{aligned}$$

and by symmetry,

$$\begin{aligned}
\frac{\partial \Phi}{\partial x_2} &= -\frac{1}{x_2^2}e^{2x_1} + e^{x_1+x_2}\left(\frac{1}{x_1} - \frac{1}{x_2} - \frac{1}{x_2^2}\right) \\
&\quad + e^{x_1}\left[\frac{1}{x_2^2} + \frac{1}{(x_1+x_2)^2}\right] \\
&\quad + e^{x_2}\left[-\frac{1}{x_1} + \frac{1}{x_1+x_2} + \frac{1}{(x_1+x_2)^2}\right] - \frac{1}{(x_1+x_2)^2}.
\end{aligned}$$

Thus,

$$\begin{aligned}
I_1 &= \frac{\partial \Phi}{\partial x_1} - \frac{\partial \Phi}{\partial x_2} \\
&= -\frac{1}{x_1^2}e^{2x_2} + \frac{1}{x_2^2}e^{2x_1} + e^{x_1+x_2}\left[\frac{2}{x_2} - \frac{2}{x_1} + \frac{1}{x_2^2} - \frac{1}{x_1^2}\right] \\
&\quad + e^{x_2}\left[\frac{1}{x_1^2} + \frac{1}{x_1} - \frac{1}{x_1+x_2}\right] + e^{x_1}\left[-\frac{1}{x_2^2} - \frac{1}{x_2} + \frac{1}{x_1+x_2}\right] \\
&\leq -\frac{1}{x_1^2}e^{2x_2} + \frac{1}{x_2^2}e^{2x_1} + e^{x_2}\left[\frac{1}{x_1^2} + \frac{1}{x_1}\right] - e^{x_1}\left[\frac{1}{x_2^2} + \frac{1}{x_2}\right] \\
&\overset{\text{sgn}}{=} -x_2^2 e^{2x_2} + x_1^2 e^{2x_1} + x_2^2(1+x_1)e^{x_2} - x_1^2(1+x_2)e^{x_1} \\
&\leq -x_2^2 e^{2x_2} + x_1^2 e^{2x_1} + x_2^2(1+x_2)e^{x_2} - x_1^2(1+x_1)e^{x_1} \\
&= -[b(x_2) - b(x_1)] \leq 0,
\end{aligned}$$

since $b(x) = x^2 e^x(e^x - 1 - x)$ is increasing.