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Stochastic comparison of parallel systems with log-Lindley distributed components

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ABSTRACT

In this paper, we study stochastic comparisons of parallel systems having log-Lindley distributed components. These comparisons are carried out with respect to reversed hazard rate and likelihood ratio ordering.

1. Introduction

In reliability optimization and life testing experiments, many times the tests are censored or truncated. For example, failure of a device during the warranty period may not be counted or items may be replaced after a certain time under a replacement policy. Moreover, test conditions, cost or other constraints may lead many reliability systems to be bounded above. This includes biological organism and human life span too. These situations result in a data set which is modeled by distributions with finite range (i.e. with bounded support) such as power function density, finite range density, truncated Weibull distribution, beta distribution, and Kumaraswamy distribution (see for example, Ghitany [7], Lai and Jones [13], Lai and Mukherjee [14], Moore and Lai [19] and Mukherjee and Islam [20]).

Beta distribution has been the most popular among the finite range distributions. Recently, Gómez et al. [8] introduced another finite range distribution, the log-Lindley (LL) distribution with shape parameter σ and scale parameter λ, written as LL(σ,λ), as an alternative to the beta distribution with the probability density function (pdf) and the cumulative distribution function (cdf) given by

$$f(x;\sigma,\lambda)=\frac{\sigma^2}{1+\lambda\sigma}(\lambda-\log x)\,x^{\sigma-1};\quad 0<x<1,\ \lambda\geq 0,\ \sigma>0, \tag{1.1}$$

and

$$F(x;\sigma,\lambda)=\frac{x^{\sigma}\,[1+\sigma\,(\lambda-\log x)]}{1+\lambda\sigma};\quad 0<x<1,\ \lambda\geq 0,\ \sigma>0 \tag{1.2}$$

respectively. This distribution has a simple expression and flexible reliability properties as compared to the beta distribution. The LL distribution exhibits bath-tub failure rates and has increasing generalized failure rate (IGFR). The distribution has useful applications in the context of inventory management, pricing and supply chain contracting problems (see, for example, Ziya et al. [28], Lariviere and Porteus [16] and Lariviere [15]), where demand distribution is required to have the IGFR property. Moreover, it has application in the actuarial context where the cdf of the distribution is used to distort the premium principle (Gómez et al. [8]). Using a single data set, Gómez et al. [8] also provides some evidence that LL distribution may provide a better fit to rates and proportions data. While LL distribution has many interesting properties and applications, ordering properties of the order statistics of this distribution under heterogeneous set-up have not been studied so far.

Order statistics play an important role in reliability optimization, life testing, operations research and many other areas. Parallel and series systems are the building blocks of many complex

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coherent systems in reliability theory. While the lifetime of a series system corresponds to the smallest order statistic $X_{1:n}$, the same of a parallel system is represented by the largest order statistic $X_{n:n}$. Although stochastic comparisons of order statistics from homogeneous populations have been studied in detail in the literature, not much work is available so far for the same from heterogeneous populations, due to its complicated nature of expressions. Such comparisons are studied with exponential, gamma, Weibull, generalized exponential, generalized Weibull or Fréchet distributed components with unbounded support. One may refer to Dykstra et al. [4], Misra and Misra [18], Zhao and Balakrishnan [24], Torrado and Kochar [23], Fang and Zhang [6], Kundu et al. [12], Kundu and Chowdhury [11], Gupta et al. [9] for more detail. Parallel systems are also compared stochastically in situations where the components are from multiple-outlier models with unbounded support. This is to be mentioned here that, a multiple-outlier model is a set of independent random variables $X_1, \ldots, X_n$ of which $X_i \overset{st}{=} X, i = 1, \ldots, n_1$ and $X_i \overset{st}{=} Y, i = n_1 + 1, \ldots, n$ where $1 \leq n_1 < n$ and $X_i \overset{st}{=} X$ means that cdf of X_i is same as that of X. In other words, the set of independent random variables $X_1, \ldots, X_n$ is said to constitute a multiple-outlier model if two sets of random variables $\left(X_1, X_2, \ldots, X_{n_1}\right)$ and $\left(X_{n_1+1}, X_{n_1+2}, \ldots, X_{n_1+n_2}\right)$ (where $n_1 + n_2 = n$), are homogeneous among themselves and heterogeneous between themselves. For more details on multiple-outlier models, readers may refer to Kochar and Xu [10], Zhao and Balakrishnan [25], Balakrishnan and Torrado [2], Zhao and Zhang [27], Kundu et al. [12], Kundu and Chowdhury [11] and the references there in. The notion of majorization (Marshall et al. [17]), as discussed in Definition 2.2 in the next section, is also essential to the understanding of the stochastic inequalities for comparing order statistics. This concept is used by El-Neweihi et al. [5] in the context of optimal component allocation in parallel-series as well as in series-parallel systems, allocation of standby in series and parallel systems. It is also used in the context of minimal repair of two-component parallel system with exponentially distributed lifetime by Boland and El-Neweihi [3].

Although significant previous research has compared series or parallel systems with infinite range distributed components, there has been little work examining similar comparisons of finite range distributed components; furthermore, all existing comparisons for finite range distributed components are specialized to the beta distribution. While Balakrishnan et al. [1] compared two 2-components beta distributed parallel systems in terms of usual stochastic order, Torrado [22] strengthened the result to likelihood ratio ordering. Zhao et al. [26] recently compared two n-components parallel systems with beta components in terms of reversed hazard rate ordering. Motivated by the usefulness of the LL distribution and the ordering properties of the beta distributed parallel systems, in this paper we compare two n-components parallel systems having heterogeneous LL distributed components in terms of reversed hazard rate order and likelihood ratio order through majorization of the parameters of the distribution. Moreover, the systems are also compared in terms of likelihood ratio order when the components are from the multiple outlier LL random variables. In this sense the paper distinguishes itself from the other few existing work in the same area. It not only uses stronger stochastic order like likelihood ratio ordering, but also compares parallel systems arising from multiple outlier LL models. The rest of the paper is organized as follows. In Section 2, we have given the required notations, definitions and some useful lemmas which have been used throughout the paper. Results related to reversed hazard rate ordering and likelihood ratio ordering between two order statistics $X_{n:n}$ and $Y_{n:n}$ are derived in Section 3.

Throughout the paper, the word increasing (resp. decreasing) and nondecreasing (resp. nonincreasing) are used interchangeably, and $\mathbb{R}$ denotes the set of real numbers $\{x : -\infty < x < \infty\}$. We also write $a \overset{sign}{=} b$ to mean that a and b have the same sign. For any differentiable function $k(\cdot)$, we write $k'(t)$ to denote the first derivative of $k(t)$ with respect to t.

2. Notations, definitions and preliminaries

Let X and Y be two absolutely continuous random variables with distribution functions $F_X(\cdot)$ and $F_Y(\cdot)$, density functions $f_X(\cdot)$ and $f_Y(\cdot)$ and reversed hazard rate functions $\tilde{r}_X(\cdot)$ and $\tilde{r}_Y(\cdot)$ respectively.

In order to compare different order statistics, stochastic orders are used for fair and reasonable comparison. In literature many different kinds of stochastic orders have been developed and studied. The following well known definitions may be obtained in Shaked and Shanthikumar [21].

Definition 2.1. Let X and Y be two absolutely continuous random variables with respective supports (l_X, u_X) and (l_Y, u_Y), where u_X and u_Y may be positive infinity, and l_X and l_Y may be negative infinity. Then, X is said to be smaller than Y in

(i) likelihood ratio (lr) order, denoted as $X \leq_{lr} Y$, if

$$\frac{f_Y(t)}{f_X(t)} \text{ is increasing in } t \in (l_X, u_X) \cup (l_Y, u_Y);$$

(ii) reversed hazard rate (rhr) order, denoted as $X \leq_{rhr} Y$, if

$$\frac{F_Y(t)}{F_X(t)} \text{ is increasing in } t \in (min(l_X, l_Y), \infty),$$

which can equivalently be written as $\tilde{r}_X(t) \leq \tilde{r}_Y(t)$ for all t;

In the following diagram we present a chain of implications of the stochastic orders, see, for instance, Shaked and Shanthikumar [21], where the definitions and usefulness of these orders can be found.

$$\begin{tikzcd} X \leq_{hr} Y \arrow[dr] & \\ X \leq_{lr} Y \arrow[u] \arrow[r] \arrow[d] & X \leq_{st} Y. \\ X \leq_{rhr} Y \arrow[ur] & \end{tikzcd}$$

It is well known that the results on different stochastic orders can be established using majorization order(s). Let I^n denote an n-dimensional Euclidean space where $I \subseteq \Re$. Further, let $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in I^n$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in I^n$ be any two real vectors with $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$ being the increasing arrangements of the components of the vector $\mathbf{x}$. The following definitions may be found in Marshall et al. [17].

Definition 2.2. The vector $\mathbf{x}$ is said to majorize the vector $\mathbf{y}$ (written as $\mathbf{x} \overset{m}{\succeq} \mathbf{y}$) if

$$\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)}, \; j = 1, \; 2, \; \ldots, n-1, \quad and \quad \sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}.$$

Definition 2.3. A function $\psi : I^n \to \Re$ is said to be Schur-convex (resp. Schur-concave) on I^n if

$$\mathbf{x} \overset{m}{\succeq} \mathbf{y} \text{ implies } \psi(\mathbf{x}) \geq (\text{resp. } \leq) \, \psi(\mathbf{y}) \; \textit{for all } \mathbf{x}, \; \mathbf{y} \in I^n.$$

Notation 2.1. *Let us introduce the following notations.*

(i) $\mathcal{D}_+ = \{(x_1, x_2, \ldots, x_n) : x_1 \geq x_2 \geq \cdots \geq x_n > 0\}$.
(ii) $\mathcal{E}_+ = \{(x_1, x_2, \ldots, x_n) : 0 < x_1 \leq x_2 \leq \cdots \leq x_n\}$.

Let us first introduce the following lemmas which will be used in the next section to prove the results.