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Convergent conic linear programming relaxations for cone convex polynomial programs

T.D. Chuong *, V. Jeyakumar

School of Mathematics and Statistics, University of New South Wales, Sydney NSW 2052, Australia

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ABSTRACT

In this paper we show that a hierarchy of conic linear programming relaxations of a cone-convex polynomial programming problem converges asymptotically under a mild well-posedness condition which can easily be checked numerically for polynomials. We also establish that an additional qualification condition guarantees finite convergence of the hierarchy. Consequently, we derive convergent semi-definite programming relaxations for convex matrix polynomial programs as well as easily tractable conic linear programming relaxations for a class of pth-order cone convex polynomial programs.

1. Introduction

Consider the cone-convex polynomial program:

$$\inf_{x\in\mathbb{R}^n}\{f(x) \mid G(x)\in -K\}, \tag{P}$$

where $f:\mathbb{R}^n\to\mathbb{R}$ is a convex polynomial, $K\subset\mathbb{R}^m$ is a closed convex cone with the vertex at the origin, and $G:\mathbb{R}^n\to\mathbb{R}^m$ is a K-convex polynomial in the sense that

$$\lambda G(x)+(1-\lambda)G(y)-G(\lambda x+(1-\lambda)y)\in K$$
$$\text{for all } x,y\in\mathbb{R}^n,\ \lambda\in[0,1],$$

$G := (G_1,\ldots,G_m)$ with G_i, $i=1,\ldots,m$, being polynomials on $\mathbb{R}^n$. The model problem of the form (P) covers a broad range of convex programming problems, including the standard convex programs with inequality constraints [3,9], convex semidefinite programs [21,6] and pth-order cone programs [3,1]. The problems of the form (P) frequently appear in robust optimization [2]. It is known, for example (cf. [2, Theorem 6.3.2]), that the robust counterpart of a convex quadratic program with a second-order cone constraint under norm-bounded uncertainty can be rewritten as a conic convex quadratic optimization problem with the positive semi-definite cone. Also, many basic control problems, such as static output feedback design problems, are modeled as conic polynomial optimization problems in terms of polynomial matrix inequalities [5,17].

Recently, an exact conic linear programming relaxation has been established in [7] for a subclass of cone-convex polynomial programs of the form (P) where the map G is K-*SOS-convex* polynomial. That study has provided a unified treatment for the semidefinite programming approximation scheme of convex polynomial programs as it covers corresponding results for problems, such as matrix SOS-convex polynomial programs [15] and SOS-convex polynomial programs with inequality constraints [8,10]. It has been derived by first developing a sum of squares polynomial representation of positivity of an SOS-convex polynomial over a conic SOS-convex inequality system with the help of a separation theorem for convex sets under a qualification condition.

In this paper we develop a new conic linear programming relaxation scheme for the cone-convex polynomial program (P) and establish its convergence. We define a hierarchy of conic linear programming relaxation problems in terms of a so-called *truncated quadratic module* and the dual cone of K, where the quadratic module involves only the objective function. It results in sum of squares relaxation problems for the problem (P) where the multiplier associated with the constraint is a constant vector. Consequently, we obtain corresponding convergent relaxations for convex matrix polynomial programs, polynomial pth-order cone convex programs and standard convex polynomial programs with inequality constraints (i.e., $K:=\mathbb{R}^m_+$). We establish the convergence of the hierarchy by employing the Putinar's Positivstellensatz [18] together with the Hahn–Banach strong separation theorem.

A convergent hierarchy of semidefinite programming (SDP) relaxations has already been given in [9, Theorem 2.1] for the

* Corresponding author.
E-mail addresses: chuongthaidoan@yahoo.com (T.D. Chuong), v.jeyakumar@unsw.edu.au (V. Jeyakumar).

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special case of (P), where $K := \mathbb{R}^m_+$ and $G := (G_1, \ldots, G_m)$. However, the multipliers associated with the constraints G_i, $i = 1, 2, \ldots, m$, of those relaxation problems are sum of squares polynomials. This results in higher degree sum of squares relaxation problems which then induce semidefinite programming relaxations of size that is often large for the present status of SDP solvers. Our result involves the truncated quadratic module (see its definition in (2.4)), where the multipliers associated with the constraints G_i, $i = 1, 2, \ldots, m$, are constants rather than sum of squares polynomials as in [9]. As a result, the present scheme has the potential to simplify the computation of the resulting semidefinite programming relaxation problems compared to the corresponding ones arising from the approach of [9].

The outline of the paper is as follows. We first show, in Section 2, that the optimal values of the conic linear programming relaxations converge asymptotically to the optimal value of the cone-convex polynomial program (P) under a mild well-posedness assumption in the sense that the feasible set of the problem (P) is nonempty and the objective function f is coercive. Consequently, we obtain corresponding results for convex matrix polynomial programs, polynomial pth-order cone convex programs, and convex polynomial programs with inequality constraints. We then show, in Section 3, that an additional qualification condition guarantees finite convergence of the hierarchy. We provide examples to illustrate how our relaxation schemes can be used to find the optimal value of the cone-convex polynomial program (P) by using the Matlab toolboxes such as CVX [4] or YALMIP [12,13].

2. Asymptotic convergence of conic linear programming relaxations

In this section, we establish an asymptotic convergence of a sequence of the conic linear programming relaxations for the cone-convex polynomial program (P). Let us start with the following qualification condition.

Well-posedness. We say that the problem (P) is well-posed if the feasible set of (P) is nonempty and the objective polynomial f is coercive, i.e., $\liminf_{\|x\|\to\infty} f(x) = +\infty$.

The first lemma provides a necessary and sufficient optimality criterion for the problem (P) without any constraint qualification. In what follows, we use the dual cone of $K \subset \mathbb{R}^m$ given by

$$K^* := \{y \in \mathbb{R}^m \mid \langle y, k\rangle \geq 0 \text{ for all } k \in K\}.$$

Lemma 2.1 (*Asymptotic Multiplier Characterization of Optimality*). *Let $\hat{x} \in \mathbb{R}^n$ be a feasible point of the well-posed problem (P). Then, $\hat{x}$ is an optimal solution of problem (P) if and only if for any $\epsilon > 0$, there exists $\lambda \in K^*$ such that*

$$f(x) + \langle \lambda, G(x)\rangle - f(\hat{x}) + \epsilon > 0, \quad \forall x \in \mathbb{R}^n. \tag{2.1}$$

Proof. [$\Longrightarrow$] Assume that $\hat{x} \in \mathbb{R}^n$ is an optimal solution of (P). Let $\epsilon > 0$ and let $C := \{x \in \mathbb{R}^n \mid G(x) \in -K\}$. As the problem (P) is well-posed, i.e., f is coercive on $\mathbb{R}^n$, which implies that the convex set

$$\Omega := \{(r, y) \in \mathbb{R}^{1+m} \mid \exists x \in \mathbb{R}^n,\ f(x) \leq r,\ y \in G(x) + K\}$$

is closed.

Then, the convex set $\widetilde{\Omega} := \Omega + \{(\epsilon - f(\hat{x}), 0)\}$ is closed as well. Since $\hat{x}$ is an optimal solution of (P), we assert that $(0, 0) \notin \widetilde{\Omega}$. The strong separation theorem (see, e.g., [14, Theorem 2.2]) guarantees that there exists $0 \neq (\lambda_0, \lambda) \in \mathbb{R} \times \mathbb{R}^m$ such that

$$\inf\Big\{\lambda_0(r + \epsilon - f(\hat{x})) + \langle \lambda, y\rangle \mid (r, y) \in \Omega\Big\} > 0.$$

This ensures that $\lambda_0 \geq 0$ and $\lambda \in K^*$. Moreover, we assert that there exists $\delta_0 > 0$ such that

$$\lambda_0(f(x) - f(\hat{x}) + \epsilon) + \langle \lambda, G(x)\rangle > \delta_0, \quad \forall x \in \mathbb{R}^n \tag{2.2}$$

due to $(f(x), G(x)) \in \Omega$ for each $x \in \mathbb{R}^n$. If $\lambda_0 = 0$, then we assert by (2.2) that $\langle \lambda, G(\hat{x})\rangle > \delta_0 > 0$. This contradicts the fact that $\hat{x}$ is a feasible point of problem (P), and thus, $\langle \lambda, G(\hat{x})\rangle \leq 0$. So, we can assume without loss of generality that $\lambda_0 = 1$ and hence (2.1) holds.

[$\Longleftarrow$] Let $\epsilon > 0$. Assume that there exists $\lambda \in K^*$ such that (2.1) holds. Let $\tilde{x} \in \mathbb{R}^n$ be an arbitrary feasible point of problem (P). It follows that $\langle \lambda, G(\tilde{x})\rangle \leq 0$ due to $\lambda \in K^*$ and $G(\tilde{x}) \in -K$. Then, $f(\tilde{x}) + \langle \lambda, G(\tilde{x})\rangle \leq f(\tilde{x})$, which together with (2.1) entails that

$$f(\tilde{x}) > f(\hat{x}) - \epsilon.$$

Since $\epsilon > 0$ was arbitrarily chosen, we conclude that $f(\hat{x}) \leq f(\tilde{x})$. Consequently, $\hat{x}$ is an optimal solution of problem (P), which completes the proof. $\square$

Denote by $\mathbb{R}[x]$ the ring of real polynomials in $x := (x_1, \ldots, x_n)$. The polynomial $f \in \mathbb{R}[x]$ is a *sum of squares* polynomial (see, e.g., [11]) if there exist polynomials $f_j \in \mathbb{R}[x]$, $j = 1, \ldots, r$ such that $f = \sum_{j=1}^r f_j^2$. The set of all sum of squares polynomials on $\mathbb{R}^n$ is denoted by Σ_n, while the set of all sum of squares polynomials on $\mathbb{R}^n$ with degree at most d is denoted by $\Sigma_{n,d}$. Given polynomials $\{g_1, \ldots, g_r\} \subset \mathbb{R}[x]$, the notation $\mathbf{M}(g_1, \ldots, g_r)$ stands for the set of polynomials generated by $\{g_1, \ldots, g_r\}$, i.e.,

$$\mathbf{M}(g_1, \ldots, g_r) := \{\sigma_0 + \sigma_1 g_1 + \cdots + \sigma_r g_r \mid \sigma_j \in \Sigma_n,\ j = 0, 1, \ldots, r\}. \tag{2.3}$$

The set $\mathbf{M}(g_1, \ldots, g_r)$ is archimedean if there exists $h \in \mathbf{M}(g_1, \ldots, g_r)$ such that the set $\{x \in \mathbb{R}^n \mid h(x) \geq 0\}$ is compact.

Lemma 2.2 (*Putinar's Positivstellensatz [18]*). *Let $f, g_j \in \mathbb{R}[x]$, $j = 1, \ldots, r$. Suppose that $\mathbf{M}(g_1, \ldots, g_r)$ is archimedean. If $f(x) > 0$ for all $x \in \{y \in \mathbb{R}^n \mid g_j(y) \geq 0,\ j = 1, \ldots, r\}$, then $f \in \mathbf{M}(g_1, \ldots, g_r)$, i.e., there exist $\sigma_j \in \Sigma_n$, $j = 0, 1, \ldots, r$, such that $f = \sigma_0 + \sum_{j=1}^r \sigma_j g_j$.*

Let $\tau \in \mathbb{R}$ be such that $\tau \geq f(\hat{x})$, where $\hat{x} \in \mathbb{R}^n$ is a feasible point of problem (P). Given $k \in \mathbb{N}$, we define the *truncated quadratic module* $\mathbf{M}_k$ generated by the polynomial $\tau - f$ as

$$\mathbf{M}_k := \{\sigma_0 + \sigma_1(\tau - f) \mid \sigma_l \in \Sigma_n, l = 0, 1, \deg(\sigma_0) \leq k, \deg(\sigma_1 f) \leq k, \deg(G_i) \leq k,\ i = 1, \ldots, m\}. \tag{2.4}$$

Conic linear programming relaxation problems. We examine a family of conic linear programming relaxation problems for the cone-convex polynomial program (P). For each $k \in \mathbb{N}$, let us consider the conic linear programming relaxation problem of (P):

$$\sup_{t\in\mathbb{R},\lambda\in\mathbb{R}^m} \Big\{t \mid f + \langle \lambda, G\rangle - t \in \mathbf{M}_k,\ \lambda \in K^*\Big\}, \tag{P$_k$}$$

where $\mathbf{M}_k$ is given by (2.4).

The first theorem shows that if the cone-convex polynomial program (P) has an optimal solution, then the optimal values of the conic linear programming relaxation problems (P$_k$) ($k \in \mathbb{N}$) converge to the optimal value of the problem (P) when the degree bound k tends to infinity.

Theorem 2.3 (*Asymptotic Convergence of Relaxations*). *Let $\bar{x} \in \mathbb{R}^n$ be an optimal solution of the well-posed problem (P). Then, we have*

$$\lim_{k\to\infty} f_k^* = f(\bar{x}),$$

where

$$f_k^* := \sup_{t\in\mathbb{R},\lambda\in\mathbb{R}^m} \Big\{t \mid f + \langle \lambda, G\rangle - t \in \mathbf{M}_k,\ \lambda \in K^*\Big\}, \quad k \in \mathbb{N}.$$