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Average value of solutions of the bipartite quadratic assignment problem and linkages to domination analysis

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ABSTRACT

We study domination analysis of algorithms for the *bipartite quadratic assignment problem*. A formula for the average objective function value of solutions is presented, whereas computing the median objective function value is shown to be NP-hard. An upper bound on the domination ratio of any polynomial time heuristic is given. Also, we show that heuristics that produce no worse than the average solutions have domination ratio at least $\frac{1}{mn}$. Heuristics with improved domination ratio are also presented.

1. Introduction

For a given $m \times n \times m \times n$ array $Q = (q_{ijk\ell})$ and $m \times n$ matrices $c = (c_{ij})$ and $d = (d_{ij})$, the *bipartite quadratic assignment problem of type 1* (QAP(B1)) is to

$$\min \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{\ell=1}^{n} q_{ijk\ell}x_{ij}y_{k\ell} + \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} + \sum_{i=1}^{m}\sum_{j=1}^{n} d_{ij}y_{ij}$$

$$\text{s.t.} \sum_{j=1}^{n} x_{ij} = 1 \quad i = 1, 2, \ldots, m, \tag{1}$$

$$\sum_{i=1}^{m} y_{ij} = 1 \quad j = 1, 2, \ldots, n, \tag{2}$$

$$x_{ij},\ y_{ij} \in \{0, 1\} \quad i = 1, \ldots, m,\ j = 1, \ldots, n.$$

Similarly, for a given $m \times m \times n \times n$ array $Q = (q_{ijk\ell})$, and $m \times m$ matrix $c = (c_{ij})$ and $n \times n$ matrix $d = (d_{ij})$, the *bipartite quadratic assignment problem of type* 2 (QAP(B2)) is to

$$\min \sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{n}\sum_{\ell=1}^{n} q_{ijk\ell}x_{ij}y_{k\ell} + \sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij}x_{ij} + \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}y_{ij}$$

$$\text{s.t.} \sum_{j=1}^{m} x_{ij} = 1 \quad i = 1, 2, \ldots, m, \tag{3}$$

$$\sum_{i=1}^{n} y_{ij} = 1 \quad j = 1, 2, \ldots, n, \tag{4}$$

$$x_{ij},\ y_{k\ell} \in \{0, 1\} \quad i, j = 1, \ldots, m,\ k, \ell = 1, \ldots, n.$$

When $m = n$, the problems QAP(B1) and QAP(B2) are the same. Furthermore, if we impose the additional restriction that $x_{ij} = y_{ij}$ for all i, j, both QAP(B1) and QAP(B2) become equivalent to the well-known *quadratic assignment problem* (QAP) [5]. Note that the constraints $x_{ij} = y_{ij}$ can be enforced simply by modifying the entries of Q, c and d without explicitly stating the constraints [23]. Hence QAP(B1) and QAP(B2) are proper generalizations of QAP.

The problems QAP(B1) and QAP(B2) were studied by Punnen and Wang in [23] where they proposed efficient heuristic algorithms to solve these problems. They also reported extensive experimental results establishing the quality of their heuristic solutions. In QAP(B1) and QAP(B2), if the constraints $x_{ij}, y_{ij} \in \{0, 1\}$ are replaced by $0 \le x_{ij}, y_{ij} \le 1$ for all i, j, we get their corresponding bilinear programming (BLP) [3,15,16] relaxations, denoted by BLP1 and BLP2, respectively. It is well known that there exists an optimal solution to the BLP which is an extreme point of the underlying convex polytope [3,15,16]. In the case of BLP1 and BLP2, the coefficient matrix of the constraints is totally unimodular and hence all extreme points are of 0–1 type. Thus, BLP1 and BLP2 are respectively equivalent to QAP(B1) and QAP(B2). Therefore, QAP(B1) and QAP(B2) can also be solved using general purpose algorithms for BLP.

QAP(B1) and QAP(B2) are known to be strongly NP-hard [23]. To the best of our knowledge, theoretical properties of these problems are not investigated thoroughly in the literature. In this

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paper, we study the complexity of QAP(B1) and QAP(B2) from the point of view of domination analysis of algorithms [2,8]. Many researchers considered such analysis for various combinatorial optimization problems [6,2,4,8–11,13,12,14,17,18,20–22,24,26,27,25,28–30]. Domination analysis is also linked to very large-scale neighborhood search [1,19] and exponential neighborhoods [7].

In this paper, we provide a closed form formula to calculate the average value of all solutions of QAP(B1) and QAP(B2) and show that there are at least $n^{m-1}m^{n-1}$ and $m^{m-1}n^{n-1}$ solutions respectively for QAP(B1) and QAP(B2) that have objective function value equal to or worse than the average objective function value of all solutions. For the standard quadratic assignment problem, although a closed form formula exists to calculate the average value of solutions, establishing non-trivial domination results is an open problem [4,13,25]. We then show that some heuristics that work well in practice, could produce solutions with objective function value worse than the average value of solutions, and we also provide simple polynomial algorithms that guarantee a solution with objective function value no worse than the average value of solutions. Unlike the average value, computing the median value of solutions for QAP(B1) and QAP(B2) is shown to be NP-hard. Further, we show that computing a solution whose objective function value is no worse than that of $n^m m^n - \lceil \frac{n}{\alpha} \rceil^{\lceil \frac{m}{\alpha} \rceil} \lceil \frac{m}{\alpha} \rceil^{\lceil \frac{n}{\alpha} \rceil}$ solutions of QAP(B1) is NP-hard for any fixed rational number $\alpha > 1$. Likewise, computing a solution whose objective function value is no worse than that of $m^m n^n - \lceil \frac{m}{\alpha} \rceil^{\lceil \frac{m}{\alpha} \rceil} \lceil \frac{n}{\alpha} \rceil^{\lceil \frac{n}{\alpha} \rceil}$ solutions of QAP(B2) is also shown to be NP-hard for any fixed rational number $\alpha > 1$.

Let $\mathcal{X}_1$ denote the set of all 0-1 $m \times n$ matrices satisfying (1), and $\mathcal{X}_2$ denote the set of 0-1 $m \times m$ matrices satisfying (3). Similarly, let $\mathcal{Y}_1$ be the set of all 0-1 $m \times n$ matrices satisfying (2) and $\mathcal{Y}_2$ be the set of all 0-1 $n \times n$ matrices satisfying (4). Also, let $\mathcal{F}_1$ and $\mathcal{F}_2$ be the sets of feasible solutions of QAP(B1) and QAP(B2), respectively. Note that $|\mathcal{F}_1| = n^m m^n$ and $|\mathcal{F}_2| = m^m n^n$. Let $M = \{1, 2, \dots, m\}$ and $N = \{1, 2, \dots, n\}$. For given cost arrays Q, c and d, and a feasible solution $(x, y) \in \mathcal{F}_1$, where $x \in \mathcal{X}_1$, $y \in \mathcal{Y}_1$, let $f_1(x, y)$ be the objective function of QAP(B1). Likewise, for $x \in \mathcal{X}_2$, $y \in \mathcal{Y}_2$, let $f_2(x, y)$ be the objective function of QAP(B2).

2. QAP(B1) and QAP(B2) as a QAP

In this section we show how every instance of QAP(B2) can be transformed to an equivalent instance of QAP. Recall that QAP can be observed as a problem of finding an optimal perfect matching on a complete bipartite graph with a quadratic objective function [5]. Similarly, every instance of QAP(B2) can be observed as a problem on the complete bipartite graphs $K_{m,m} = (V_1 \cup V_2, V_1 \times V_2)$ and $K_{n,n} = (V_3 \cup V_4, V_3 \times V_4)$, where vertices are denoted by $V_1 = \{u_1, \dots, u_m\}$, $V_2 = \{v_1, \dots, v_m\}$, $V_3 = \{w_1, \dots, w_n\}$ and $V_4 = \{z_1, \dots, z_n\}$, as follows. Every $x \in \mathcal{X}_2$ corresponds to m edges (u_i, v_j) of $K_{m,m}$ for which $x_{ij} = 1$. That is, $x \in \mathcal{X}_2$ corresponds to a set of m edges where exactly one is incident to each vertex of V_1, see (3). Similarly, $y \in \mathcal{Y}_2$ corresponds to a set of n edges of $K_{n,n}$, where exactly one is incident to each vertex of V_4, see (4). See Fig. 1 for an example. Then to every edge (u_i, v_j) of $K_{m,m}$ and (w_k, z_ℓ) of $K_{n,n}$, costs c_{ij} and $d_{k\ell}$ are assigned, respectively. Furthermore, cost $q_{ijk\ell}$ is assigned to every pair of edges (u_i, v_j), (w_k, z_ℓ).

To create a corresponding bipartite graph instance of QAP, we need to ensure that the edges that correspond to $(u_i, v_{j'})$, $(u_i, v_{j''})$ (which simultaneously can appear in a feasible solution of QAP(B2)) are not adjacent in the created QAP instance. And similarly, we need to ensure that the edges that correspond to $(w_{i'}, z_j)$, $(w_{i''}, z_j)$ are not adjacent in the created QAP instance. To do so, for each $u_i \in V_1$ we create a copy of $K_{m,m}$ denoted by $K^i_{m,m}$, and for each $z_\ell \in V_4$ we create a copy of $K_{n,n}$ denoted by $K^\ell_{n,n}$. In every $K^i_{m,m}$, linear costs of edges that are not incident to u_i are changed to 0. Analogously, in every $K^\ell_{n,n}$, linear costs of edges that are not incident to z_ℓ are changed to 0. Now the QAP instance is created by concatenating $K^i_{m,m}$'s, for $i = 1, \dots, m$, and $K^\ell_{n,n}$'s, for $\ell = 1, \dots, n$, into one bipartite graph that we denote by G, see Fig. 2. Lastly, to insure that in every feasible solution of the created QAP instance, all edges that appear are within the same $K^i_{m,m}$ or $K^\ell_{n,n}$, we define the linear costs of edges that are not within the same $K^i_{m,m}$ or $K^\ell_{n,n}$ to be equal to some large value L.

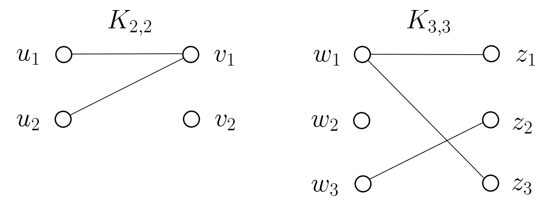

Fig. 1. The bipartite graph visualization of a feasible solution of QAP(B2) with $x_{11} = x_{21} = y_{11} = y_{32} = y_{13} = 1$.

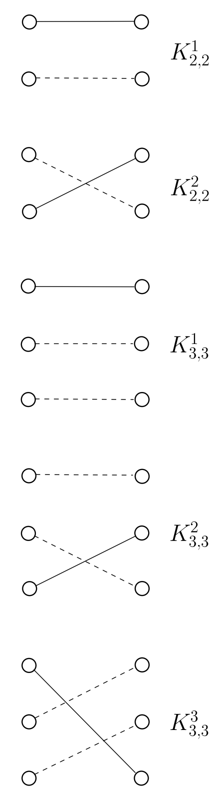

Fig. 2. A feasible solution of a QAP instance created from QAP(B2) which correspond to the solution in Fig. 1. Solid edges are the only one that contribute to the objective function value.

To complete the transformation, we need to formally describe all the costs associated with G. That is, we need to define the linear cost matrix $\bar{c}$ and the quadratic cost array $\bar{Q}$ of the corresponding QAP. Linear costs are described above, and are formally given by:

$$\bar{c}_{(i-1)m+i,(i-1)m+j} = c_{ij} \quad \forall\, i, j = 1, \dots, m$$

and

$$\bar{c}_{m^2+(j-1)n+i,m^2+(j-1)n+j} = d_{ij} \quad \forall\, i, j = 1, \dots, n.$$

Furthermore, $\bar{c}_{ij} = L$ when $i, j \le m^2$ and $\lfloor (i-1)/m \rfloor \ne \lfloor (j-1)/m \rfloor$, or $i, j > m^2$ and $\lfloor (i-m^2-1)/n \rfloor \ne \lfloor (j-m^2-1)/n \rfloor$. Also, $\bar{c}_{ij} = L$ if $i \le m^2 < j$ or $j \le m^2 < i$. Lastly, remaining $\bar{c}_{ij}$'s are set to be 0. It remains to define the quadratic costs which are given by

$$\bar{q}_{(i-1)m+i,(i-1)m+j,m^2+(\ell-1)n+k,m^2+(\ell-1)n+\ell} = q_{ijk\ell}$$