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A note on link formation and network stability in a Hotelling game

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ABSTRACT

We develop a model to examine the link formation and the stability of networks in a Hotelling-type oligopoly. We find that with two firms, the link formation depends on the degree of vertical differentiation regardless of the degree of horizontal differentiation, while, with a greater number of firms, link formation occurs when firms feature high horizontal differentiation but low vertical differentiation.

1. Introduction

We draw on the literature on the networks of collaboration in oligopoly markets [7,6,4,2] to seek out the incentives of firms to form pair-wise links in a Hotelling-type game embedding both vertical and horizontal product differentiation. In particular, we study the conditions under which a pair-wise link is stable, namely the conditions under which firms participate in a network taking advantage of collaboration opportunities with others. The model is able to cover all cases of the network stability, starting with two firms ($N = 2$) and then extending the analysis to the more general case $N \geq 3$.

With the simple case of two firms we show that link formation just depends on the degree of vertical differentiation.

In the more general case of $N \geq 2$ the main result is that link formation requires minimal vertical differentiation and maximal horizontal differentiation as a result of less severe price competition. We show how it is possible to draw a closed form solution when the network is made up of two and three firms.

As a matter of fact, a general analysis of asymmetric networks turns out to be very complex. Indeed, to draw some results, many seminal papers have restricted the analysis to the simplest case of three or even two firms. Some examples may include, among others, [5,1,7], and [2]. Further contributions on firm cooperation in two and three firm games are also in [3,10]. [5] studies the endogenous formation of the cooperation structure between players, showing that several equilibrium refinements give rise to the formation of complete cooperation structures; differently, in the three player game, only a pair of agents forms a link. [1] examines a generalised Hotelling game where firms choose location and then compete à la Bertrand. He obtains an analytical solution of the three-firm case proving that the equilibrium profit of the centre firm considerably exceeds the corner firms' profits. [7] considers a three-firm market for a homogeneous good showing that asymmetric networks may substantially alter the market structure by causing significant disparities between firms. However, they also show that this is not necessarily detrimental from a social perspective. Three-firm asymmetric networks are also studied in [2]; differently from [7], the authors develop a model with absorptive capacity showing that for small levels of spillovers, R&D investment is higher in denser networks and the complete network is socially efficient. The methodology used in this model draws on this literature that makes use of a three-firm case as an intermediate step before studying the more general and complex case with $N \geq 3$. However, differently from the current literature, we study network formation under the hypothesis that firms are asymmetric. It allows us to truly explore the implications of differentiated firms in forming networks. To the best of our knowledge this paper is the first attempt to study the link formation and the stability of networks in an asymmetric Hotelling-type game. We show that with high vertical but low horizontal differentiation strategically stable networks are incomplete (namely empty, partially connected or some star networks) as a result of the high costs of forming links borne by high-quality firms and the more severe price competition. Indeed, firms with higher quality have no incentive to form links because they run the risk of making their lower quality partners better at their own expense. On the contrary, we find that when firms feature low vertical but high horizontal differentiation they benefit from forming links and denser networks are thus pair-wise stable.

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The remainder of the paper is articulated as follows. Section 2 provides a short description of the model; in Section 2.1 we analyse, as a benchmark, the simple case with two firms ($N = 2$); then we extend the analysis to the cases with three firms ($N = 3$) (Section 2.2) as an intermediate step to deal with to the more complex and general case with $N \geq 3$ firms (Section 2.3). Finally, Section 3 concludes.

2. The model

Let us consider $N \geq 2$ oligopolistic-type firms which are vertically and horizontally differentiated taking part in a two-stage game. In the first stage, firms might form collaboration links aimed at lowering marginal costs of production while, in the second stage, firms compete by setting prices. Furthermore, network formation is assumed to follow [8], namely it is not strategically modelled.

Let $I_i \in [0, 1]$, $i = 1, \ldots, N$ denote the fixed physical location of firm (product) i on the [0, 1] interval (the Hotelling line) with $I_{i-1} < I_i < I_{i+1}$. Turning to the demand side, it is assumed that a representative consumer has preferences over the firm goods that read as follows:

$$u = r + \theta_i - \tau (z - I_i)^2 - p_i, \tag{1}$$

where u is the utility for one unit of good i having quality $\theta_i \in \left[\underline{\theta}, \bar{\theta}\right]$; r is the willingness to pay, τ measures the unit transportation cost and p_i the price of product i. Consumers are uniformly distributed on the Hotelling line and $z \in [0, 1]$ is the consumer's location along this line.

The indifferent consumer between firms i and j is thus:

$$z_{ij} = \frac{\theta_i - \theta_j}{2\tau \left(I_j - I_i\right)} + \frac{I_j + I_i}{2} + \frac{p_j - p_i}{2\tau \left(I_j - I_i\right)}, \tag{2}$$

and the demand functions for the generic internal firm i and the two corner firms 1 and N are respectively $D_i = z_{i,i+1} - z_{i-1,i}$, $D_1 = z_{12}$ and $D_N = 1 - z_{N-1,N}$. Turning to the second stage of the game in which firms compete by prices, we define the profit function of the firm i as follows:

$$\pi_i = (p_i - c_i) D_i, \quad i \in \{1, \ldots, N\}, \tag{3}$$

where c_i is the cost for producing one unit of good i. Let us assume that the unit cost function c_i has the following form:

$$c_i = c - l \sum_{j \neq i} s_{ij}\theta_j, \quad s_{ij} \in \{0, 1\}, \quad l \in [0, 1], \tag{4}$$

where s_{ij} is a binary variable representing the pair-wise relationship between firms i and j; when firms i and j are linked then $s_{ij} = 1$, while $s_{ij} = 0$ means no link. Furthermore, l is the exogenous rate of the knowledge spillover, namely describing the proportion of knowledge that spills over from firm j to firm i.

Firms could reduce their costs by forming alliances based on knowledge sharing. The knowledge that firm i can absorb from firm j is represented by the quality of the collaborating firm j.

More formally, the firm locations $\{I_i\}_{i=1}^N$, the quality levels $\{\theta_i\}_{i=1}^N$ and the profit functions $\{\pi_i\}_{i=1}^N$ define a Hotelling game. We thus aim to calculate the Hotelling equilibrium $\left\{p_i^*\right\}_{i=1}^N$ and then the maximised profit functions $\left\{\pi_i^*\right\}_{i=1}^N$ to find and discuss the conditions under which stable links and thus pair-wise stable networks occur, according to the following definition:

Definition ([8]). A network g is pair-wise stable if, for any pair of firms i, j, the following conditions hold:

1. if $s_{ij} = 1$ then $\pi_i^*(g) \geq \pi_i^*(g - s_{ij})$ and $\pi_j^*(g) \geq \pi_j^*(g - s_{ij})$;
2. if $s_{ij} = 0$ and $\pi_i^*(g + s_{ij}) > \pi_i^*(g)$ then $\pi_j^*(g + s_{ij}) < \pi_j^*(g)$. □

Equivalently, a network g is pair-wise stable if $\frac{\partial \pi_i^*}{\partial s_{ij}} \geq 0$ and $\frac{\partial \pi_j^*}{\partial s_{ij}} \geq 0$ at the same time for any pair of firms i, j. The network $g + s_{ij}$ is thus obtained by replacing $s_{ij} = 0$ in network g with $s_{ij} = 1$; by the same token, the network $g - s_{ij}$ is obtained by severing an existing link between firms i and j.

2.1. The benchmark case of two firms

We briefly discuss the case of two firms as a starting point to study the more general situation in which more than two firms operate. To be sure, it is simply an exercise of comparative statics of the Hotelling equilibrium with respect to the costs, but it is nonetheless useful to shed light on the main mechanisms at work in the model. Deriving profit function (3) with respect to p_1 and p_2 we obtain equilibrium prices and demands, $p_1^* = \frac{\theta_1 - \theta_2}{3} + \tau \frac{(I_2 - I_1)(I_2 + I_1 + 2)}{3} + \frac{2}{3}(c - ls_{12}\theta_2) + \frac{1}{3}(c - ls_{12}\theta_1)$, $p_2^* = \frac{\theta_2 - \theta_1}{3} + \tau \frac{(I_2 - I_1)(4 - I_2 - I_1)}{3} + \frac{2}{3}(c - ls_{12}\theta_1) + \frac{1}{3}(c - ls_{12}\theta_2)$, $D_1 = \frac{I_2 + I_1}{6} + \frac{1}{3} + \frac{\theta_1 - \theta_2}{6\tau(I_2 - I_1)} + \frac{ls_{12}(\theta_2 - \theta_1)}{6\tau(I_2 - I_1)}$ and $D_2 = 1 - D_1$ from which, substituting into $\pi_i = (p_i - c_i) D_i \ \forall i = 1, 2$, we obtain the maximised profit functions of the two firms that read:

$$\pi_1^* = \left[\frac{\theta_1 - \theta_2}{3} + \frac{\tau}{3}(I_2 - I_1)(I_2 + I_1 + 2) + \frac{1}{3} ls_{12}(\theta_2 - \theta_1)\right] \times \left[\frac{I_2 + I_1}{6} + \frac{1}{3} + \frac{\theta_1 - \theta_2}{6\tau(I_2 - I_1)} + \frac{ls_{12}(\theta_2 - \theta_1)}{6\tau(I_2 - I_1)}\right], \tag{5}$$

$$\pi_2^* = \left[\frac{\theta_2 - \theta_1}{3} + \frac{\tau}{3}(I_2 - I_1)(4 - I_2 - I_1) - \frac{1}{3} ls_{12}(\theta_2 - \theta_1)\right] \times \left[1 - \frac{I_2 + I_1}{6} - \frac{1}{3} - \frac{\theta_1 - \theta_2}{6\tau(I_2 - I_1)} - \frac{ls_{12}(\theta_2 - \theta_1)}{6\tau(I_2 - I_1)}\right]. \tag{6}$$

According to the Jackson–Wolinsky definition of pair-wise stability, firms 1 and 2 will be linked if $\frac{\partial \pi_1^*}{\partial s_{12}} \geq 0$ and $\frac{\partial \pi_2^*}{\partial s_{12}} \geq 0$ at the same time. It is thus easy to detect a condition under which the pair-wise stability condition is fulfilled. The following proposition summarises the point.

Proposition 1. *If $\theta_2 = \theta_1$ then the two firms will form a link.*

Proof. (See Supplementary material, Appendix A.) □

It is worth noticing that in this highly simplified case, the link formation does not depend on the degree of (low or high) horizontal differentiation but on vertical differentiation. Obviously, in this case if two firms form a link under the above conditions ($\theta_2 = \theta_1$) they also yield a stable complete network. The intuition behind this result is that in the two firm case, each firm has a unique direct competitor, independently of the degree of horizontal differentiation; hence, in deciding about link formation, the two firms only take into account the difference in product quality.

2.2. The three-firm case

We now discuss the case of three firms. The possible network structures that three firms might give rise to are depicted in Figure 1 (see Supplementary material, Appendix E) while the first order conditions, equilibrium prices and maximised profit functions are reported in the Appendix B (see Supplementary material).

The results are summarised in the following propositions.

Proposition 2. (a) *If $I_3 - I_1 \neq 1$, then:*
–Firms 1 and 2 are linked, i.e. $s_{12} = 1$, if $\frac{I_3 - I_2}{I_3 - I_1} \leq \frac{\theta_1}{\theta_2} \leq \frac{2I_3 + I_2 - 3I_1}{2(I_3 - I_1)}$;
–Firms 1 and 3 are linked, i.e. $s_{13} = 1$, if $\frac{I_3 - I_2}{3I_3 - I_2 - 2I_1} \leq \frac{\theta_1}{\theta_3} \leq \frac{2I_3 + I_2 - 3I_1}{I_2 - I_1}$;