Engineering Analysis with Boundary Elements 36 (2012) 600–605

Contents lists available at SciVerse ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

# A hybrid finite difference and moving least square method for elasticity problems

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ARTICLE INFO

Article history:
Received 24 May 2011
Accepted 6 October 2011
Available online 15 December 2011

Keywords:
Finite difference method
Meshless method
Point collocation method
Elastodynamic

ABSTRACT

In this paper, a novel hybrid finite difference and moving least square (MLS) technique is presented for the two-dimensional elasticity problems. A new approach for an indirect evaluation of second order and higher order derivatives of the MLS shape functions at field points is developed. As derivatives are obtained from a local approximation, the proposed method is computationally economical and efficient. The classical central finite difference formulas are used at domain collocation points with finite difference grids for regular boundaries and boundary conditions are represented using a moving least square approximation. For irregular shape problems, a point collocation method (PCM) is applied at points that are close to irregular boundaries. Neither the connectivity of mesh in the domain/boundary or integrations with fundamental/particular solutions is required in this approach. The application of the hybrid method to two-dimensional elastostatic and elastodynamic problems is presented and comparisons are made with the boundary element method and analytical solutions.



## 1. Introduction

Meshless approximations have gained much popularity since Nayroles et al. [1] proposed the diffuse element method. Later, Belyschko et al. [2] and Liu et al. [3] proposed the element-free Galerkin method and reproducing kernel particle methods, respectively. A feature of these methods is that they do not require a structured grid and are hence meshless. Recently, Atluri and Zhu [4,5] presented a family of Meshless methods based on the Local weak Petrov–Galerkin formulation (MLPGs) for arbitrary partial differential equations with MLS approximation. MLPG is reported to provide a rational basis for constructing meshless methods with a greater degree of flexibility. Local Boundary Integral Equation method (LBIE) with MLS and polynomial radial function has been developed by Sladek et al. [6–8] for the boundary value problems in anisotropic non-homogeneous media. Wen and Aliabadi [9–12] and Li et al. [13] have extended the meshless approach to problems in fracture mechanics and woven composites.

In the last decade, developments of the radial basis functions (RBF) as a truly meshless method has drawn the attention of many investigators (see Golberg et al. [14]). Hardy [15] and Hon and Mao [16] used multiquadric interpolation method for solving linear differential equation. Li et al. [17] compared the method of fundamental solutions (MFS) and dual reciprocity method (DRM), by the use of radial basis functions.

As an alternative approach, finite point method was proposed by Oñate et al. [18] for elasticity problems on the basis of weighted least-square procedure. Zhang et al. [19] developed a least-square collocation meshfree method to improve the solution accuracy. They demonstrated the application of a new moving least-square technique, which differs from original MLS to elasticity and crack problems. For crack growth problems investigated by Lee and Yoon [20], the inherent advantage of not involving a mesh generation is retained. In addition, the speed of computing derivatives of meshfree approximation was accelerated by a derivative-approximating technique.

Finite Difference Method [21,22] (FDM) belongs to the strong-form methods and the formulation procedure is relatively simple and straightforward compared with the meshless weak-form method. Like other strong form method, FDM suffers from the instability problems particularly for structural problems with stress boundary conditions. Radial bases function interpolation based on finite difference method was introduced by Liu et al. [23]. By incorporating the radial point interpolation into the classical finite difference approach with a least square technique, the resultant set of algebraic equations were solved more efficiently and accurately than using a typical PCM. For the point collocation method, the difficulty of determination of high order derivatives of shape functions has been overcome by an indirect scheme proposed by Wen and Aliabadi [24].

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doi:10.1016/j.enganabound.2011.10.002

In this paper, MLS approximation is introduced into the finite difference method using the same finite difference grid net, i.e., the boundary conditions using MLS are applied at the boundary collocation points, improved point collocation method is applied to grid points near irregular boundaries and the classical FDM to domain net points. Unlike the radial bases function, there are no field points to be introduced either on the boundary or inside the domain.

## 2. Point collocation method

### 2.1. MLS interpolation

Consider a domain $\Omega$ with boundary $\Gamma$ containing a sub-domain $\Omega_y$ as shown in Fig. 1. The sub-domain is in neighbourhood of a point $\mathbf{y}$ and is considered as the domain of MLS approximation for the trial function at $\mathbf{y}$. This domain is called the support domain to an arbitrary point $\mathbf{y}$. To interpolate the distribution of function $u$ in the sub-domain $\Omega_y$ over a number of randomly distributed nodes $\{\mathbf{y}_i\}=\{y_{1i},y_{2i}\}$, $i=1,2,\dots,n_y$, we have the approximation of function $u$ at the point $\mathbf{y}$ as

$$u(\mathbf{y})=\mathbf{p}(\mathbf{y})^T\mathbf{a}(\mathbf{y}) \tag{1}$$

where $\mathbf{a}(\mathbf{y})$ is a vector of unknown coefficients and $\mathbf{p}(\mathbf{y})^T=\{p_1(\mathbf{y}), p_2(\mathbf{y}),\dots,p_m(\mathbf{y})\}$ is a complete monomial basis, $m$ denotes the number of terms in the basis, i.e. for two-dimensional problems

$$\mathbf{p}^T(\mathbf{y})=\{1,y_1,y_2\},\quad \text{linear basis}\;\; m=3; \tag{2}$$

$$\mathbf{p}^T(\mathbf{y})=\{1,y_1,y_2,y_1^2,y_1y_2,y_2^2\},\quad \text{quadraticbasis}\;\; m=6. \tag{3}$$

The unknown coefficient vector $\mathbf{a}(\mathbf{y})$ is determined by minimising $L_2$ norm with a weighted function $\mathbf{w}(\mathbf{y},\mathbf{x})$ as

$$J(\mathbf{a})=\sum_{i=1}^{n_y} w_i(\mathbf{y},\mathbf{x}_i)[\mathbf{p}^T(\mathbf{x}_i)\mathbf{a}(\mathbf{y})-\hat{u}_i(\mathbf{x}_i)]^2 \tag{4}$$

where $\mathbf{x}_i$ denotes the position vector of node $i$ in the support domain, $w_i(\mathbf{y},\mathbf{x}_i)$ is the weight function associated with the node $i$ with $w_i(\mathbf{y},\mathbf{x}_i)>0$ for all $\mathbf{x}$ in the support domain and $\hat{u}_i(\mathbf{x}_i)$ is the fictitious nodal values, but in general not the values of the unknown trial function at the nodes, $\overline{u}_i(\mathbf{x}_i)$. The stationary value of $J(\mathbf{a})$ with respect to $\mathbf{a}(\mathbf{y})$ leads to a linear relation between the coefficient vector $\mathbf{a}(\mathbf{y})$ and the vector of fictitious node values $\hat{\mathbf{u}}$ as follows:

$$\mathbf{A}(\mathbf{y},\mathbf{x})\mathbf{a}(\mathbf{y})=\mathbf{B}(\mathbf{y},\mathbf{x})\hat{\mathbf{u}} \tag{5}$$

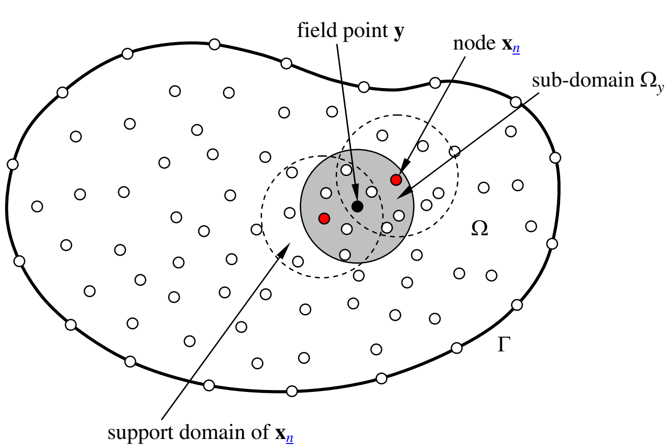


**Fig. 1.** Sub-domain $\Omega_y$ for MLS approximation of the field point $\mathbf{y}$ and the support area around node $\mathbf{x}^n$.

where matrices $\mathbf{A}(\mathbf{y},\mathbf{x})$ and $\mathbf{B}(\mathbf{y},\mathbf{x})$ are defined by

$$\mathbf{A}(\mathbf{y},\mathbf{x})=\mathbf{p}^T\mathbf{w}\mathbf{p}=\sum_{i=1}^{n_y} w_i(\mathbf{y},\mathbf{x}_i)\mathbf{p}(\mathbf{x}_i)\mathbf{p}^T(\mathbf{x}_i) \tag{6}$$

$$\mathbf{B}(\mathbf{y},\mathbf{x})=\mathbf{p}^T\mathbf{w}=[w_1(\mathbf{y})\mathbf{p}(\mathbf{x}_1),w_2(\mathbf{y})\mathbf{p}(\mathbf{x}_2),\dots,w_{n(\mathbf{y})}(\mathbf{y})\mathbf{p}(\mathbf{x}_{n(\mathbf{y})})]. \tag{7}$$

The MLS approximation is well defined only when the matrix $\mathbf{A}(\mathbf{y})$ in Eq. (5) is non-singular. A necessary condition to satisfy this requirement is that at least $m$ weight functions are non-zero (i.e. $n_y>m$) for each sample point $\mathbf{y}\in\Omega$ and that the nodes in $\Omega_y$ will not by arranged in a special pattern such as on a straight line. Solving linear equations in Eq. (5) for coefficients $\mathbf{a}(\mathbf{y})$ gives following relation:

$$u(\mathbf{y})=\Phi^T(\mathbf{y},\mathbf{x})\hat{\mathbf{u}}=\sum_{i=1}^{n(\mathbf{y})}\phi_i(\mathbf{y},\mathbf{x}_i)\hat{u}_i \tag{8}$$

where

$$\Phi^T(\mathbf{y},\mathbf{x})=\mathbf{p}^T(\mathbf{y})\mathbf{A}^{-1}(\mathbf{y},\mathbf{x})\mathbf{B}(\mathbf{y},\mathbf{x}) \tag{9}$$

or

$$\phi_i(\mathbf{y},\mathbf{x})=\sum_{j=1}^{m}p_j(\mathbf{y})[\mathbf{A}^{-1}(\mathbf{y},\mathbf{x})\mathbf{B}(\mathbf{y},\mathbf{x})]_{ji} \tag{10}$$

Usually $\phi_i(\mathbf{y},\mathbf{x})$ is called the shape function of the MLS approximation corresponding to the nodal point $\mathbf{x}$. The support area of the nodal point $\mathbf{x}$ is taken to be a circle of radius $d_y$ centred at $\mathbf{x}$ (same size of local sub-domain centred at field point $\mathbf{y}$). The selection of the radius $d_y$ is important in the MLS approximation because it determines the range of the interaction between the degrees of freedom defined at the considered nodes. The size of the support domain ($r$) should be sufficiently large to cover the nodes in the domain of definition hence ensuring the regularity of the matrix $\mathbf{A}$. In the numerical process, the radius $d_y$ will be determined by the minimum number of $n_y$ in the sub-domain. A fourth order spline type weight function is defined as

$$w_i(\mathbf{y},\mathbf{x}_i)=\begin{cases}1-6\left(\frac{r}{d_y}\right)^2+8\left(\frac{r}{d_y}\right)^3-3\left(\frac{r}{d_y}\right)^4, & 0\le r\le d_y\\ 0, & d_y\le r\end{cases} \tag{11}$$

where $r=|\mathbf{y}-\mathbf{x}_i|$. As the matrices in the shape function $\mathbf{A}^{-1}(\mathbf{y},\mathbf{x})$ and $\mathbf{B}(\mathbf{y},\mathbf{x})$ in Eq. (10) are functions of field points and nodal positions in the support domain, the determination of high order derivatives of shape functions with respect to the field point $\mathbf{y}$ will become more complicated in the numerical process.

### 2.2. Direct technique for MLS shape function's derivatives

The partial derivatives of shape function can be obtained from Eq. (8) by a straightforward differentiation, using MLS, as

$$u_{,k}(\mathbf{y})=\Phi^T_{,k}(\mathbf{y},\mathbf{x})\hat{\mathbf{u}}=\sum_{i=1}^{n_y}\phi_{i,k}(\mathbf{y},\mathbf{x}_i)\hat{u}_i \tag{12}$$

where $(\;)_{,k}$ denotes $\partial(\;)/\partial y_k$ and

$$\phi_{i,k}(\mathbf{y},\mathbf{x}_i)=\sum_{i=1}^{n_y}[p_{i,k}(\mathbf{A}^{-1}\mathbf{B})+p_i(\mathbf{A}^{-1}_{,k}\mathbf{B}+\mathbf{A}^{-1}\mathbf{B}_{,k})] \tag{13}$$

As $\mathbf{A}^{-1}\mathbf{A}=\mathbf{I}$, the derivative of the inverse of matrix $\mathbf{A}$ with respect to $y_k$ is given by

$$\mathbf{A}^{-1}_{,k}=-\mathbf{A}^{-1}\mathbf{A}_{,k}\mathbf{A}^{-1} \tag{14}$$