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A multiwavelet Galerkin boundary element method for the stationary Stokes problem in 3D

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ABSTRACT

In this paper, a multiwavelet Galerkin boundary element method is presented for the fast solution of the stationary Stokes problem in three dimensions. Piecewise linear discontinuous multiwavelet bases are constructed on each patch of piecewise smooth surface individually, which allow easy and efficient evaluation of the matrix entries. Because of the use of the multiwavelets, the system matrix can be compressed to $O(N)$ (N denotes the number of unknowns) nonzero entries without compromising the order of convergence as for the conventional Galerkin boundary element method. Numerical results of two test samples are given to demonstrate the availability of the present method.

1. Introduction

The Stokes problem has been usually applied to model incompressible creeping flows where the fluid Reynolds number is very low. The problem has been studied extensively by boundary element method (BEM). The BEM is often more convenient than the traditional finite element method (FEM), since the corresponding equations are formulated on the boundary, which satisfy the incompressibility constraint. However, the system matrices of boundary element equations are computed densely due to the nonlocal nature of the boundary integral operators, which is the main disadvantage of the BEM compared with FEM that leads to sparse matrices. This drawback makes inconvenience to apply the BEM to the large-scale problems. However, many methods for the fast solution of BEM have been developed in the last decades. Prominent examples for such methods are the fast multipole method [1–3], the multi-level BEM [4,5], the panel clustering [6], H-matrices [7] and the adaptive cross approximation [8]. Furthermore, wavelet Galerkin BEM (GBEM) [9–29] was introduced and successfully applied to many practical problems in the last years.

In Ref. [29], we have described a GBEM using Alpert multiwavelets proposed in Ref. [10] for solving the two-dimensional Stokes problem. The multiwavelets not only have short supports and high order of vanishing moments but also are simply piecewise polynomials which allow easy and efficient evaluation of the matrix entries. In this paper, we will present a multiwavelet GBEM (MGBEM) for solving the three-dimensional Stokes problem, in which the multiwavelets are constructed on each patch individually. The multiwavelets have higher order of vanishing moment compared with Haar wavelets [17]; hence, we can obtain a sparser system matrix containing $O(N)$ nonzero entries by using two step compression strategies. What's more, the multiwavelets are simpler piecewise polynomials compared with Spline wavelets [12,18,19,22], therefore, we can calculate the matrix entries in shorter times.

We consider the stationary Stokes problem with Dirichlet boundary condition

$$\begin{cases} -\mu\Delta u_i(\mathbf{x})+\frac{\partial p(\mathbf{x})}{\partial x_i}=0, & \mathbf{x}\in\Omega \text{ or } \Omega', \\ \nabla\cdot\mathbf{u}(\mathbf{x})=\sum_{i=1}^{3}\dfrac{\partial u_i(\mathbf{x})}{\partial x_i}=0, & \mathbf{x}\in\Omega \text{ or } \Omega', \\ u_i(\mathbf{x})=g_i(\mathbf{x}), & \mathbf{x}\in\Gamma, \quad i=1,2,3 \end{cases} \tag{1}$$

where Ω is an open bounded domain in R^3 of points $\mathbf{x}=(x_1,x_2,x_3)$, its boundary Γ is assumed to be piecewise smooth, the complement of $\overline{\Omega}=\Omega+\Gamma$ is denoted by Ω', and $\mathbf{n}=(n_1,n_2,n_3)$ denotes the unit exterior normal to Γ. The unknowns are the velocity $\mathbf{u}=(u_1,u_2,u_3)$ and pressure p of the viscous incompressible fluid filled in Ω or Ω', here the kinematic viscosity μ is constant, $\mathbf{g}=(g_1,g_2,g_3)$ is a given function on Γ.

The solution $(\mathbf{u},p)$ of problem (1) can be expressed in the form of the simple layer potentials [30,31,32]

$$\begin{cases} u_j(\mathbf{y})=\sum_{i=1}^{3}\int_\Gamma t_i(\mathbf{x})U_{ij}(\mathbf{x},\mathbf{y})dS_\mathbf{x}, & j=1,2,3,\ \mathbf{y}\in\mathbf{R}^3, \\ p(\mathbf{y})=\sum_{i=1}^{3}\int_\Gamma t_i(\mathbf{x})P_i(\mathbf{x},\mathbf{y})dS_\mathbf{x}, & \mathbf{y}\in R^3-\Gamma, \end{cases} \tag{2}$$

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where $\mathbf{t}=(t_1,t_2,t_3)$ stands for the vectorial density to be determined and (U_{ij},P_i) is the fundamental solution of Stokes equation

$$\begin{cases} U_{ij}(\mathbf{x},\mathbf{y}) = \frac{1}{8\pi\mu}\left(\frac{\delta_{ij}}{|\mathbf{x}-\mathbf{y}|}+\frac{(x_i-y_i)(x_j-y_j)}{|\mathbf{x}-\mathbf{y}|^3}\right), \\ P_i(\mathbf{x},\mathbf{y}) = \frac{x_i-y_i}{4\pi|\mathbf{x}-\mathbf{y}|^3},\ i,j=1,2,3, \end{cases} \tag{3}$$

where δ_{ij} is the Kronecker symbol.

Using the boundary condition of problem (1) one obtains the boundary integral formulation

$$g_j(\mathbf{y}) = \sum_{i=1}^{3}\int_\Gamma t_i(\mathbf{x})U_{ij}(\mathbf{x},\mathbf{y})dS_{\mathbf{x}},\ j=1,2,3,\ \mathbf{y}\in\Gamma, \tag{4}$$

which defines a continuous mapping $\mathbf{g}\in\mathbf{U}(\Gamma)\rightarrow\mathbf{t}\in(H^{-1/2}(\Gamma))^3$ and is equivalent to the following Galerkin variational problem [30,32]

$$\begin{cases} \text{find } \mathbf{t}\in\mathbf{T}(\Gamma),\ \text{ such that } \forall\mathbf{t}'\in\mathbf{T}(\Gamma), \\ b(\mathbf{t},\mathbf{t}') = \langle\mathbf{g},\mathbf{t}'\rangle_{L^2(\Gamma)}, \end{cases} \tag{5}$$

where $\mathbf{T}(\Gamma)=(H^{-1/2}(\Gamma))^3/\Sigma$, Σ denotes an equivalent relation: $\mathbf{t}\sim\mathbf{t}'$ if and only if $\mathbf{t}-\mathbf{t}'=\lambda\mathbf{n}, \lambda\in R$, $b(\mathbf{t},\mathbf{t}')$ and $\langle\mathbf{g},\mathbf{t}'\rangle_{L^2(\Gamma)}$ are given by

$$\begin{cases} b(\mathbf{t},\mathbf{t}') = \sum_{i,j=1}^{3}\int_\Gamma\int_\Gamma t_i(\mathbf{x})t_j'(\mathbf{y})U_{ij}(\mathbf{x},\mathbf{y})dS_{\mathbf{x}}dS_{\mathbf{y}}, \\ \langle\mathbf{g},\mathbf{t}'\rangle_{L^2(\Gamma)} = \sum_{j=1}^{3}\int_\Gamma g_j(\mathbf{y})t_j'(\mathbf{y})dS_{\mathbf{y}}. \end{cases} \tag{6}$$

Here, considering that $\mathbf{t}$ is unique in the sense of differing by an additive constant vector proportional to $\mathbf{n}$, we add an equation for obtaining a unique solution

$$\int_\Gamma \mathbf{t}\cdot\mathbf{n}dS = 0. \tag{7}$$

Thus, $\mathbf{t}$ can be uniquely determined by combining Eqs. (5) and (7).

The rest of this paper is outlined as follows. Section 2 gives the construction of the multiwavelets on piecewise smooth surface in three dimensions. Then, the MGBEM is described in Section 3. In Section 4, two numerical examples are given to demonstrate the availability of the present method. Finally, Section 5 contains some conclusions.

2. Multiwavelet bases

In the conventional GBEM, the problem (5) is approximated in a finite dimensional subspace with single scale bases. This leads to a dense linear system. To overcome this drawback, the problem (5) will be approximated in a nested family of finite dimensional trial spaces $V_J(\Gamma)\subset V_{J+1}(\Gamma)$ which consist of piecewise polynomial functions in local coordinates and are spanned by multiwavelet bases.

To obtain the spaces $V_J(\Gamma)$, we assume that the boundary Γ is given as a parametric surface consisting of smooth patches. More precisely, Γ can be partitioned into N_Γ patches $\Gamma_e(e=1,2,\cdots,N_\Gamma)$ which are smooth images of the reference element $E=\{\boldsymbol{\xi}=(\xi_1,\xi_2)\in R^2{:}0<\xi_1<1, 0<\xi_2<1-\xi_1\}$, i.e., there exist bijective mappings $\boldsymbol{X}_e(e=1,2,\ldots,N_\Gamma)$ which are analytic in E, such that $\Gamma_e=\boldsymbol{X}_e(E)$. The partition $\{\Gamma_e\}_{e=1}^{N_\Gamma}$ is assumed to be regular, i.e., for $e\neq e'$, $\Gamma_e\cap\Gamma_{e'}$ is either empty or a vertex or a common edge.

As shown in Fig. 1, by successively dividing the reference element E into 4^J $(J\geq 0)$ congruent triangles $\{E_k^J\}_{k=1}^{4^J}$, we define a space $S_J(E)$ of piecewise polynomial functions

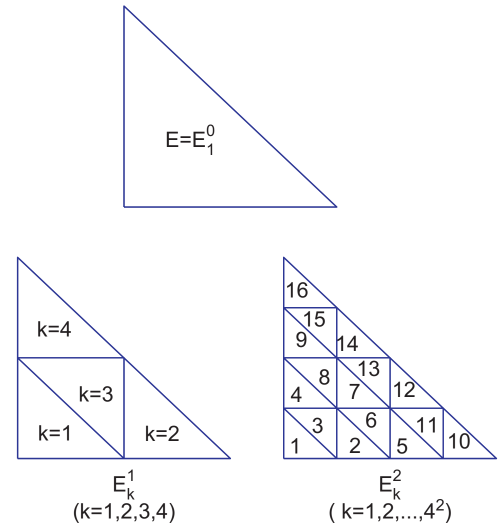

Fig. 1. Reference element.

$$S_J(E) = \{v\in L^2(E): v|_{E_k^J}\in P_m,\quad k=1,\ldots,4^J\}. \tag{8}$$

Let $M=(m+1)(m+2)/2$ denote the dimension of P_m, then the space $S_J(E)$ has dimension $s_J=M4^J$. It is apparent that

$$S_0(E)\subset S_1(E)\subset\cdots\subset S_J(E)\subset\cdots. \tag{9}$$

For $J=1,2,3,\ldots$, we define the $r_J=3M4^{J-1}$ dimensional space $K_J(E)$ to be the orthogonal complement of $S_{J-1}(E)$ in $S_J(E)$, i.e., $S_J(E)=S_{J-1}(E)\oplus K_J(E)$, so we inductively obtain the multiscale decomposition

$$S_J(E) = S_0(E)\oplus K_1(E)\oplus\cdots\oplus K_J(E). \tag{10}$$

The polynomial space P_m on the reference element E can be spanned by the conventional monomial base $\{\varphi_i(\boldsymbol{\xi})\}_{i=1}^M$. For $J=0,1,2,\ldots$, we define functions $\varphi_{J,k,i}{:}E\rightarrow R$ by

$$\varphi_{J,k,i} = \begin{cases} \varphi_i\circ\mu_k^J, & \boldsymbol{\xi}\in E_k^J, \\ 0, & \boldsymbol{\xi}\,\overline{\in}\,E_k^J, \end{cases} \tag{11}$$

with $k=1,2,\ldots,4^J, i=1,2,\ldots,M$. μ_k^J denotes the affine transformation mapping from E_k^J to E. Then the set $\{\varphi_{J,k,i}\}$ forms the single scale base of the space $S_J(E)$.

Suppose that function set $\{\psi_i\}_{i=1}^{r_1}$ forms a base for $K_1(E)$. Since $K_1(E)$ is orthogonal to $S_0(E)$, the multiwavelet bases $\{\psi_i\}_{i=1}^{r_1}$ have vanishing moments of $m+1$ order

$$\langle\psi_i(\boldsymbol{\xi}),\boldsymbol{\xi}^{\boldsymbol{\upsilon}}\rangle_{L^2(E)} = \int_E\psi_i(\boldsymbol{\xi})\boldsymbol{\xi}^{\boldsymbol{\upsilon}}d\boldsymbol{\xi} = 0 \tag{12}$$

in which $\boldsymbol{\upsilon}=(\upsilon_1,\upsilon_2)\in\mathbf{N}_0^2$, $|\boldsymbol{\upsilon}|=\upsilon_1+\upsilon_2<m+1$. For $J=1,2,3,\ldots$, we define functions $\psi_{J,k,i}{:}E\rightarrow R$ by

$$\psi_{J,k,i} = \begin{cases} \psi_i\circ\mu_k^{J-1}, & \boldsymbol{\xi}\in E_k^{J-1}, \\ 0, & \boldsymbol{\xi}\,\overline{\in}\,E_k^{J-1}, \end{cases} \tag{13}$$

with $k=1,2,\ldots,4^{J-1}$, $i=1,2,\ldots,r_1$. Then the set $\{\psi_{J,k,i}\}$ forms the base of the space $K_J(E)$. It is obvious that the base functions have also vanishing moments of $m+1$ order

$$\int_E\psi_{J,k,i}(\boldsymbol{\xi})\boldsymbol{\xi}^{\boldsymbol{\upsilon}}d\boldsymbol{\xi} = \int_{E_k^J}(\psi_i\circ\mu_k^J)\boldsymbol{\xi}^{\boldsymbol{\upsilon}}d\boldsymbol{\xi} = \int_E\psi_i(\boldsymbol{\xi})\boldsymbol{\xi}^{\boldsymbol{\upsilon}}d\boldsymbol{\xi} = 0,\ |\boldsymbol{\upsilon}|<m+1. \tag{14}$$

Now via the parametric representations of the boundary Γ, we define the single scale bases and multiwavelet bases on the