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Boundary element analysis for viscoelastic solids containing interfaces/holes/cracks/inclusions

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ABSTRACT

With the aid of the elastic–viscoelastic correspondence principle, the boundary element developed for the linear anisotropic elastic solids can be applied directly to the linear anisotropic viscoelastic solids in the Laplace domain. Green's functions for the problems of two-dimensional linear anisotropic elastic solids containing holes, cracks, inclusions, or interfaces have been obtained analytically using Stroh's complex variable formalism. Through the use of these Green's functions and the correspondence principle, special boundary elements in the Laplace domain for viscoelastic solids containing holes, cracks, inclusions, or interfaces are developed in this paper. Subregion technique is employed when multiple holes, cracks, inclusions, and interfaces exist simultaneously. After obtaining the physical responses in Laplace domain, their associated values in time domain are calculated by the numerical inversion of Laplace transform. The main feature of this proposed boundary element is that no meshes are needed along the boundary of holes, cracks, inclusions and interfaces whose boundary conditions are satisfied exactly. To show this special feature by comparison with the other numerical methods, several examples are solved for the linear isotropic viscoelastic materials under plane strain condition. The results show that the present BEM is really more efficient and accurate for the problems of viscoelastic solids containing interfaces, holes, cracks, and/or inclusions.

1. Introduction

Viscoelastic materials exhibit a time and rate dependence that is completely absent in the elastic materials. Due to the inclusion of time as an independent variable, the available exact analytical solutions have been obtained only for a few simplified problems. Thus, to study the mechanical behavior of viscoelastic solids, the numerical approaches such as finite element method (FEM) and boundary element method (BEM) are normally needed. The main advantages of BEM are the reduction of the problem dimension by one and the exact satisfaction of certain boundary conditions for particular problems if their associated fundamental solutions are embedded in boundary element formulation. Generally, there are three different approaches to linear viscoelastic analysis by BEM. The first formulates a BEM in Laplace transform domain and obtain the solution in time domain by numerical inversion [1,2]. The second formulates a BEM directly in time domain [3–5]. Although the second approach looks more direct and efficient, but the lack of fundamental solutions in time domain restricts the applicability of time domain BEM approach. To combine the advantages of the previous two approaches, a mixed BEM was proposed by Schanz [6], which can solve the problem in time domain but rely on the fundamental solutions in Laplace domain.

Through the use of correspondence principle, the viscoelastic solids can be effectively treated in Laplace domain. To take advantage of the available fundamental solutions for the defects or interfaces in anisotropic elastic materials [7], in this paper we choose the first approach, i.e., the transformed BEM to treat the problems of viscoelastic solids containing defects such as holes, cracks or inclusions, or interfaces. Using the subregion technique [8], the problems with simultaneous existence of multiple holes, cracks, inclusions, and interfaces can also be treated without too much extra works. The main feature of this proposed method is that no meshes are needed along the boundary of defects and interfaces whose boundary conditions are satisfied exactly, which means that the present approach should be more efficient and accurate. To show this special feature, several examples considering interfaces, holes, cracks, and/or inclusions are illustrated in this paper. For the purpose of comparison, some examples are taken from those treated by the other BEM methods such as a circular elastic inclusion in an isotropic viscoelastic solid simulated by Kelvin model [9], an elliptical hole in an isotropic viscoelastic solid simulated by generalized Kelvin model [1], and a center-cracked plate in an isotropic viscoelastic solid under plane strain condition [10]. As to the interface problems, the

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comparison is made with the results obtained by the commercial finite element software ANSYS. To show the applicability to the problems with multiple holes, cracks, inclusions, and interfaces, an example for the estimation of effective Young's modulus of fiber-reinforced polymers is done and compared with the analytical solutions presented in the literature [11,12]. All these results show that the present BEM is really efficient and accurate for the problems of viscoelastic solids containing interfaces, holes, cracks, and/or inclusions.

2. Linear anisotropic viscoelasticity

In a fixed rectangular coordinate system x_i, $i=1,2,3$, let u_i, σ_{ij}, and ε_{ij} be, respectively, the displacement, stress, and strain. The constitutive laws for the linear anisotropic viscoelastic materials, the strain–displacement relations for the small deformations, and the equilibrium equations for static loading conditions can be written as [13]

$$\sigma_{ij}(t)=C_{ijkl}(t)^{*}d\varepsilon_{kl}(t),\quad \varepsilon_{ij}(t)=\tfrac{1}{2}\{u_{i,j}(t)+u_{j,i}(t)\},\quad \sigma_{ij,j}(t)=0, \tag{2.1}$$

where $i,j,k,l=1,2,3$, and the repeated indices imply summation; a comma stands for differentiation; $C_{ijkl}(t)$ is the elastic stiffness tensor whose components are also known to be the *relaxation functions* of the viscoelastic materials, and the symmetry of stress and strain imply $C_{ijkl}(t)=C_{jikl}(t)=C_{ijlk}(t)$. In the first equation of (2.1), the notation of the Stieltjes convolution has been used, i.e.,

$$C_{ijkl}(t)^{*}d\varepsilon_{kl}(t)=\int_{-\infty}^{t}C_{ijkl}(t-\tau)d\varepsilon_{kl}(\tau). \tag{2.2}$$

If the applied strain history begins at $t=0$ with a non-zero initial value, and $\varepsilon_{ij}=0$ for $t<0$, (2.2) can be further reduced to

$$C_{ijkl}(t)^{*}d\varepsilon_{kl}(t)=C_{ijkl}(t)\varepsilon_{kl}(0)+\int_{0}^{t}C_{ijkl}(t-\tau)\frac{\partial\varepsilon_{kl}(\tau)}{\partial\tau}d\tau. \tag{2.3}$$

Taking the Laplace transform of (2.1) gives

$$\check{\sigma}_{ij}(s)=s\check{C}_{ijkl}(s)\check{\varepsilon}_{kl}(s),\quad \check{\varepsilon}_{ij}(s)=\tfrac{1}{2}\{\check{u}_{i,j}(s)+\check{u}_{j,i}(s)\},\quad \check{\sigma}_{ij,j}(s)=0, \tag{2.4}$$

where s is the transform variable and the Laplace transform $\check{f}(s)$ of $f(t)$ is defined as

$$\check{f}(s)=\int_{0}^{\infty}f(t)e^{-st}dt. \tag{2.5}$$

Eq. (2.4) is identical to the basic equations of linear anisotropic elasticity, which means that the viscoelastic solutions in the Laplace transform domain can be obtained directly from the solutions of the corresponding elastic problems with the replacement of the elastic stiffness tensor C_{ijkl} by $s\check{C}_{ijkl}(s)$, if the boundary of a viscoelastic body is invariant with time. This statement is the so-called *correspondence principle* between linear elasticity and linear viscoelasticity [13,14] and is applicable to anisotropic viscoelastic materials.

2.1. Stroh formalism for viscoelasticity in Laplace transform domain

By using the correspondence principle and Stroh formalism for two-dimensional linear anisotropic elasticity [7,15], the general solutions satisfying the 15 partial differential equations (2.4) can be written as

$$\check{\mathbf{u}}=2\mathrm{Re}\{\mathbf{Af}(z)\},\quad \check{\boldsymbol{\phi}}=2\mathrm{Re}\{\mathbf{Bf}(z)\}, \tag{2.6a}$$

where

$$\check{\mathbf{u}}=\begin{Bmatrix}\check{u}_1\\ \check{u}_2\\ \check{u}_3\end{Bmatrix},\quad \check{\boldsymbol{\phi}}=\begin{Bmatrix}\check{\phi}_1\\ \check{\phi}_2\\ \check{\phi}_3\end{Bmatrix},\quad \mathbf{f}(z)=\begin{Bmatrix}f_1(z_1)\\ f_2(z_2)\\ f_3(z_3)\end{Bmatrix},\quad z_\alpha=x_1+\mu_\alpha x_2,$$

$$\mathbf{A}=[\mathbf{a}_1\quad \mathbf{a}_2\quad \mathbf{a}_3],\quad \mathbf{B}=[\mathbf{b}_1\quad \mathbf{b}_2\quad \mathbf{b}_3], \tag{2.6b}$$

and Re stands for the real part. $\check{\mathbf{u}}$ and $\check{\boldsymbol{\phi}}$ are the displacement and stress function vectors in Laplace transform domain, respectively, and $\check{\phi}_i$, $i=1,2,3$ are related to the stresses in Laplace transform domain by

$$\check{\sigma}_{i1}=-\check{\phi}_{i,2},\quad \check{\sigma}_{i2}=\check{\phi}_{i,1}. \tag{2.7}$$

$\mathbf{f}(z)$ is a function vector composed of three holomorphic complex functions $f_\alpha(z_\alpha)$, $\alpha=1,2,3$, which will be determined by the satisfaction of boundary conditions. μ_α and ($\mathbf{a}_\alpha$, $\mathbf{b}_\alpha$) are the material eigenvalues and eigenvectors which can be determined by the following eigenrelations:

$$\mathbf{N}\boldsymbol{\xi}=\mu\boldsymbol{\xi}, \tag{2.8a}$$

where $\mathbf{N}$ is a 6×6 fundamental elasticity matrix and $\boldsymbol{\xi}$ is a 6×1 column vector defined by

$$\mathbf{N}=\begin{bmatrix}\mathbf{N}_1 & \mathbf{N}_2\\ \mathbf{N}_3 & \mathbf{N}_1^T\end{bmatrix},\quad \boldsymbol{\xi}=\begin{Bmatrix}\mathbf{a}\\ \mathbf{b}\end{Bmatrix}, \tag{2.8b}$$

and

$$\mathbf{N}_1=-\mathbf{T}^{-1}\mathbf{R}^T,\quad \mathbf{N}_2=\mathbf{T}^{-1}=\mathbf{N}_2^T,\quad \mathbf{N}_3=\mathbf{R}\mathbf{T}^{-1}\mathbf{R}^T-\mathbf{Q}=\mathbf{N}_3^T, \tag{2.8c}$$

where the superscript T denotes the transpose of a matrix. $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{T}$ are three 3×3 real matrices defined by the elastic constants as

$$Q_{ik}=s\check{C}_{i1k1}(s),\quad R_{ik}=s\check{C}_{i1k2}(s),\quad T_{ik}=s\check{C}_{i2k2}(s),\quad i,k=1,2,3. \tag{2.8d}$$

Although the Laplace parameter s can be a complex variable, in this paper during the inversion of the Laplace transform only the function values related to the *real-positive* parameter s is considered. In this sense, all the numerical results show that $s\check{C}_{ijkl}$ is positive definite. Although the rigorous mathematical proof about the positive definiteness is not provided in this paper, it seems that it can be proved through the consideration that the strain energy at every time stage of the viscoelastic solids is always positive. Thus, with a real-positive Laplace parameter s the material eigenvalues μ_α obtained from the eigen-relation (2.8) cannot be real, and μ_α occurs as three pairs of complex conjugates. In the general solution (2.6), the material eigenvalues μ_α and material eigenvectors ($\mathbf{a}_\alpha$,$\mathbf{b}_\alpha$) have been arranged to be $\mu_{\alpha+3}=\bar{\mu}_\alpha$, $\mathrm{Im}(\mu_\alpha)>0$, and $\mathbf{a}_{\alpha+3}=\bar{\mathbf{a}}_\alpha$, $\mathbf{b}_{\alpha+3}=\bar{\mathbf{b}}_\alpha$, $\alpha=1,2,3$, where an overbar denotes the complex conjugate and Im stands for the imaginary part. Moreover, in the general solution (2.6), the material eigenvalues are assumed to be distinct and their associated eigenvectors are independent of each other. For the cases that the material eigenvalues are repeated so that their associated eigenvectors are not independent of each other, the general solution (2.6) should be modified or one may introduce a small perturbation in the values of material properties to avoid the problem of degeneracy [7,15].

An alternative and more direct way to calculate the material eigenvalues and eigenvectors is through the following characteristic equation [7,15,16]

$$l_4(\mu)l_2(\mu)-l_3^{\,2}(\mu)=0, \tag{2.9a}$$

where

$$l_2(\mu)=\mu q_5-q_4,\quad l_3(\mu)=\mu^2q_2+q_2-\mu q_6,\quad l_4(\mu)=\mu^2p_1+p_2-\mu p_6, \tag{2.9b}$$

and

$$p_j(\mu)=s[\mu^2\check{S}^r_{j1}+\check{S}^r_{j2}-\mu\check{S}^r_{j6}],\quad q_j(\mu)=s[\mu\check{S}^r_{j5}-\check{S}^r_{j4}],\quad j=1,2,4,5,6. \tag{2.9c}$$