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A new iterative integral formulation for semilinear equations based on the generalized quasilinearization theory

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ABSTRACT

This paper presents a new iterative integral approach for solving semilinear equations. The integral formulation is derived based on the generalized quasilinearization theory in which nonlinear equations are replaced by a set of iterative linear equations. An advantage of the new formulation is that its convergence is guaranteed under a given condition and the convergence rate can be quadratic. The effectiveness of the new approach has been demonstrated on several examples of the nonlinear Poisson type. Comparisons with some existing methods and a study of the convergence rate have also been conducted in this work.

1. Introduction

The boundary element method has established itself as one of the most powerful numerical methods for solving various linear problems. Extension to nonlinear problems has been actively pursued due to the advantages that integral-based approaches can offer. A common treatment of the nonlinear term is to regard it as a "pseudo" body force and the integral formulation is formed based on the fundamental solution of the linear operator of the equation. As such, a volume integral containing the nonlinear term inevitably results. Various methods, such as domain discretization free methods [1–6], and cell-based direct integration schemes [7–10], have been proposed and developed to efficiently and accurately evaluate volume integrals. In the first category of the methods, volume integrals are either eliminated using a particular solution if it is available [1,2], or are transformed into boundary integrals using interpolation based on radial basis functions as in, for example, the dual reciprocity method (DRM) [3,4] and the multiple-reciprocity method (MRM) [5], or using sub-domain quasilinearization as in the quasilinear boundary element method (QBEM) [6]. The second class of the methods uses volume cells to directly evaluate the nonlinear volume integrals. Efforts have been focused on efficient cell generation. Methods such as the auxiliary domain subtraction method [7,8] and the grid-based integral approach [9,10] have been developed. In [11], a brief review and a comparison of some aforementioned methods are presented.

Due to the nonlinear nature of the problem, an iterative scheme must be adopted at a certain stage during the problem solving process, independent of the method employed (except for the particular solution method). A popular iterative scheme is the direct successive-substitution method, which is essentially a variation of the Picard iteration method. In this method, an initial guess of the unknown function is assumed at the beginning of the iteration and the nonlinear term, in an integral form, is evaluated based on the assumed value. The unknown function is then updated by solving the corresponding linear equation. The convergence of this method depends on the level of nonlinearity as well as the choice of the initial guess. Often, for problems with severe nonlinearity this method fails to converge. To achieve a better convergence, an implicit scheme can be adopted in which the unknown quantities at the boundary nodes and interior nodes are solved simultaneously, usually by the Newton–Raphson method. The drawback of this method is that additional equations must be supplied in order to close the system, which leads to a much larger discretized system. In addition, although the convergence of this approach in general is better than the direct successive-substitution scheme, it is nevertheless still not guaranteed. In this paper a different iterative scheme is employed and a corresponding integral formulation is derived with the aim of achieving a fast convergence rate without the need of increasing the size of the discretized system. This scheme is based on the generalized quasilinearization theory [12,13]. Under certain circumferences, a quadratic convergence rate is guaranteed even when the nonlinearity is severe.

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The paper is organized as follows. A brief review of the classical boundary element method (BEM) and the dual reciprocity boundary element method (DRBEM) for solving semilinear equations using the direct successive-substitutive scheme is presented first, followed by the description of the new iterative integral formulation and its connection with the generalized quasilinearization theory. Results are then presented to demonstrate the effectiveness of the formulation through several illustrative examples together with a comparison with some other methods. In Section 4, the convergence rate of the formulation is discussed and conclusions are given in the last section.

2. Methodologies

Without losing generality, the nonlinear Poisson equation shown in Eq. (1) is chosen as an illustrative example of the semilinear equations considered in this work

$$\begin{aligned} &\nabla^2 u = f(\mathbf{x},u), \quad u \in \Omega \\ &Bu = \phi, \qquad\quad u \in \partial\Omega \end{aligned}, \tag{1}$$

where u is the unknown function, $\mathbf{x}$ is the spatial variable and Ω is the problem domain. The boundary condition is described by $Bu = p(\mathbf{x})u + q(\mathbf{x})(\partial u/\partial n) = \phi$ with n being the outward normal of the boundary $\partial\Omega$ and p and q are two general funtions of $\mathbf{x}$.

2.1. Review of some existing integral-based approaches: the classical BEM and the DRBEM

In a classical boundary element approach for solving Eq. (1), the integral representation is formulated using the fundamental solution of the Laplace operator as

$$c(\xi)u(\xi) + \int_{\partial\Omega} \frac{\partial G(\mathbf{x},\xi)}{\partial n(\mathbf{x})} u(\mathbf{x}) dS(\mathbf{x}) - \int_{\partial\Omega} G(\mathbf{x},\xi) \frac{\partial u(\mathbf{x})}{\partial n(\mathbf{x})} dS(\mathbf{x}) = -\int_{\Omega} G(\mathbf{x},\xi) f(u(\mathbf{x}),\mathbf{x}) dV(\mathbf{x}), \tag{2}$$

where $c(\xi)$ is the self-term, defined as $c(\xi) = \begin{cases} 1, & \xi \in \Omega \\ 0, & \xi \notin (\Omega \cup \partial\Omega) \\ \alpha, & \xi \in \partial\Omega \end{cases}$, and the second integral should be interpreted in the sense of Cauchy Principal Value when ξ is on the boundary. In Eq. (2), $n(\mathbf{x})$ is the unit outward normal vector at the field point $\mathbf{x}$, ξ is the evaluation/source point and $G(\mathbf{x}, \xi)$ is the Green's function of the Laplace operator. The volume integral at the right-hand side of Eq. (2) involves the unknown function u at the interior of the domain. Applying the direct successive-substitution method, an iterative scheme is constructed as

$$\begin{aligned} &c(\xi)u^{(m+1)}(\xi) + \int_{\partial\Omega} \frac{\partial G(\mathbf{x},\xi)}{\partial n(\mathbf{x})} u^{(m+1)}(\mathbf{x}) dS(\mathbf{x}) - \int_{\partial\Omega} G(\mathbf{x},\xi) \frac{\partial u^{(m+1)}(\mathbf{x})}{\partial n(\mathbf{x})} dS(\mathbf{x}) \\ &\quad = -\int_{\Omega} G(\mathbf{x},\xi) f(u^{(m)}(\mathbf{x}),\mathbf{x}) dV(\mathbf{x}), \quad m = 0,1,2,\cdots, \end{aligned} \tag{3}$$

with $u^{(0)}$ being the initial guess. The iteration terminates when the difference in u between two consecutive iterations is smaller than a pre-set tolerance.

In the dual reciprocity boundary element approach, the nonlinear term f is first approximated by a set of radial basis functions, $\phi_k(\mathbf{x})$, which are chosen so that functions $g_k(\mathbf{x})$ satisfying $\nabla^2 g_k = \phi_k$ can be found. The interpolation reads

$$f(u,\mathbf{x}) \approx \sum_{k=1}^{NT} \alpha_k \phi_k, \tag{4}$$

where α_k is the interpolation coefficient and NT is the total number of interpolation nodes including both boundary and interior nodes. The nonlinear Poisson equation is then approximated as

$$\nabla^2 u \approx \sum_{k=1}^{NT} \alpha_k \nabla^2 g_k, \tag{5}$$

and is solved using the standard boundary element method with the integral formulation shown as

$$\begin{aligned} &c(\xi)u(\xi) + \int_{\partial\Omega} \frac{\partial G(\mathbf{x},\xi)}{\partial n(\mathbf{x})} u(\mathbf{x}) dS(\mathbf{x}) - \int_{\partial\Omega} G(\mathbf{x},\xi) \frac{\partial u(\mathbf{x})}{\partial n(\mathbf{x})} dS(\mathbf{x}) \\ &\quad \approx \sum_{k=1}^{NT} \alpha_k \left[c(\xi) g_k(\xi) + \int_{\partial\Omega} \frac{\partial G(\mathbf{x},\xi)}{\partial n(\mathbf{x})} g_k(\mathbf{x}) dS(\mathbf{x}) \right] \\ &\quad\quad - \sum_{k=1}^{NT} \alpha_k \left[\int_{\partial\Omega} G(\mathbf{x},\xi) \frac{\partial g_k(\mathbf{x})}{\partial n(\mathbf{x})} dS(\mathbf{x}) \right]. \end{aligned} \tag{6}$$

Note the interpolation coefficient α_k depends on u at each node, more specifically $\alpha_k = \sum_{l=1}^{NT} P_{kl} f(u_l, \mathbf{x}_l)$, where P is the interpolation matrix and l denotes the l-th interpolation node. The application of the direct successive-substitution method results in the following iterative scheme:

$$\begin{aligned} &c(\xi)u^{(m+1)}(\xi) + \int_{\partial\Omega} \frac{\partial G(\mathbf{x},\xi)}{\partial n(\mathbf{x})} u^{(m+1)}(\mathbf{x}) dS(\mathbf{x}) - \int_{\partial\Omega} G(\mathbf{x},\xi) \frac{\partial u^{(m+1)}(\mathbf{x})}{\partial n(\mathbf{x})} dS(\mathbf{x}) \\ &\quad \approx \sum_{k=1}^{NT} (\alpha_k)^{(m)} c(\xi) g_k(\xi) + \sum_{k=1}^{NT} (\alpha_k)^{(m)} \int_{\partial\Omega} \frac{\partial G(\mathbf{x},\xi)}{\partial n(\mathbf{x})} g_k(\mathbf{x}) dS(\mathbf{x}) \\ &\quad\quad - \sum_{k=1}^{NT} (\alpha_k)^{(m)} \int_{\partial\Omega} G(\mathbf{x},\xi) \frac{\partial g_k(\mathbf{x})}{\partial n(\mathbf{x})} dS(\mathbf{x}), \end{aligned} \tag{7}$$

with $(\alpha_k)^{(m)} = \sum_{l=1}^{NT} P_{kl} f\left(u_l^{(m)}, \mathbf{x}_l\right)$.

2.2. New iterative integral formulation

The final iterative equations, Eq. (3) and (7), derived from the classical BEM and the DRBEM, respectively, are in fact two different schemes for solving the following set of linear equations

$$\begin{aligned} &\nabla^2 u^{(m+1)} = f(\mathbf{x},u^{(m)}), \quad m = 0,1,2,\cdots \\ &Bu^{(m+1)} = \phi. \end{aligned} \tag{8}$$

This is essentially a variation of the Picard method for solving the original nonlinear Poisson equation. It is well known that the convergence of Eq. (8) is not guaranteed. It depends on the level of nonlinearity as well as the initial guess. A numerical example will be illustrated in Section 3.

A similar set of linear equations but with a better convergence can be constructed based on the generalized quasilinearization theory proposed by Bellman and Kalaba [12] and later further developed by Lakshmikantham and Koksal [13]. It reads

$$\begin{aligned} &\nabla^2 u^{(m+1)} - \left.\frac{\partial f}{\partial u}\right|_{u^{(m)}} \cdot u^{(m+1)} = f(\mathbf{x},u^{(m)}) - \left.\frac{\partial f}{\partial u}\right|_{u^{(m)}} \cdot u^{(m)} \\ &Bu^{(m+1)} = \phi \quad m = 0,1,2,\cdots. \end{aligned} \tag{9}$$

The advantage of this set of iterative equations is that under certain conditions, the sequence $\{u^{(m)}\}$ obtained from Eq. (9) converges to the solution of the original nonlinear Poisson equation, $\nabla^2 u = f(\mathbf{x},u)$, monotonically for a wide range of initial guesses and the convergence is quadratic. For example, if $\partial^2 f/\partial u^2 \geq 0$, the sequence of $\{u^{(m)}\}$ defined in Eq. (9) converges for any initial guess satisfying $\nabla^2 u^{(0)} \leq f(\mathbf{x},u^{(0)})$ and $Bu^{(0)} \geq \phi$. If $\partial^2 f/\partial u^2 \leq 0$, the sequence of $\{u^{(m)}\}$ converges for any initial guess satisfying $\nabla^2 u^{(0)} \geq f(\mathbf{x},u^{(0)})$ and $Bu^{(0)} \leq \phi$. In both cases, the convergence rate is quadratic. In cases when f is neither a convex nor a concave function of u, it is