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The method of fundamental solutions for a time-dependent two-dimensional Cauchy heat conduction problem

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ABSTRACT

We investigate an application of the method of fundamental solutions (MFS) to the time-dependent two-dimensional Cauchy heat conduction problem, which is an inverse ill-posed problem. Data in the form of the solution and its normal derivative is given on a part of the boundary and no data is prescribed on the remaining part of the boundary of the solution domain. To generate a numerical approximation we generalize the work for the stationary case in Marin (2011) [23] to the time-dependent setting building on the MFS proposed in Johansson and Lesnic (2008) [15], for the one-dimensional heat conduction problem. We incorporate Tikhonov regularization to obtain stable results. The proposed approach is flexible and can be adjusted rather easily to various solution domains and data. An additional advantage is that the initial data does not need to be known a priori, but can be reconstructed as well.

1. Introduction

The Cauchy problem is a classical inverse problem, where temperature and normal heat flux data is missing from a part of the boundary, and is recovered from data overspecified on the remaining part of the boundary. For this Cauchy problem we assume that the boundary shape is known (including the part of the boundary with missing data), and thermal diffusivities, conductivities, etc. are also known. The problem is ill-posed in the Hadamard sense, see [11], since even if a solution exists it will not depend continuously on the given data. Due to the lack of continuous dependence, regularization is required to obtain stable numerical results.

To solve the above Cauchy problem we apply the method of fundamental solutions (MFS) which is applicable when the fundamental solution of the homogeneous partial differential equation (PDE) governing the problem in question is known. The MFS is a collocation method and therefore has advantages due to its relative simplicity and being computationally inexpensive compared to methods which require mesh generation in the solution domain (FEM and FDM) or numerical integration over the boundary (BEM). The MFS was introduced as a numerical method by Mathon and Johnston in [25], and over recent decades has become increasingly popular for both direct and inverse problems, see the survey papers [8,9,20] for details. The MFS has been primarily applied to elliptic PDEs, see [2,4], and it has also been applied to Cauchy problems for Helmholtz-type equations in [24] and recently to steady-state heat conduction Cauchy problems in [23]. In [23] it was demonstrated that the MFS worked well and was easy to adjust to various solution domains. Thus it is natural to extend [23] to the time-dependent setting. In [15] the MFS was applied to the one-dimensional heat conduction problem, and we extend this work to the application of the MFS for the two-dimensional time-dependent Cauchy heat conduction problem, for which, to the best of the authors' knowledge, there are considerably fewer results than in the stationary case. We note that various formulations of the MFS for the parabolic heat equation have been given in [5,10,22,30,33], note though that for those there are in general no denseness results available for the approximation.

We start the work in Section 2 by stating the notation that will be used in the paper and give a mathematical formulation of the problem. In Section 3, we construct the MFS solution and give details of its implementation and Tikhonov regularization. In Section 4, we present numerical results for different domains (including rectangular, epitrochoidal and teardrop shaped domains) and various solutions (including examples with singular functions and with no analytic solution) as well as noisy boundary data.

2. Mathematical formulation of the Cauchy problem

Let $\mathbf{x}=(x_1,x_2)\in\mathbb{R}^2$, D be a two-dimensional heat conducting body, with piecewise smooth boundary Γ, and closure $\overline{D}=D\cup\Gamma$. For $T>0$ is a fixed real number, then the extensions of the

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domain D and boundary Γ in time are given by $D_T = D \times (0,T]$, and $\Gamma_T \times (0,T]$. Let $\Gamma^{(1)}$ be an open arc of Γ and put $\Gamma^{(2)} = \Gamma \backslash \Gamma^{(1)}$. Define $\Gamma_T^{(i)} = \Gamma^{(i)} \times (0,T]$, $i=1,2$. We assume that Cauchy data is given on $\Gamma_T^{(1)}$. Let, as usual, $\nabla = (\partial_{x_1}, \partial_{x_2})$ and $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2$.

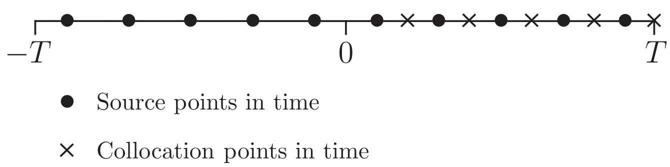

Fig. 1. Representation of the placement of collocation and source points in time ($M_1 = M = 5$).

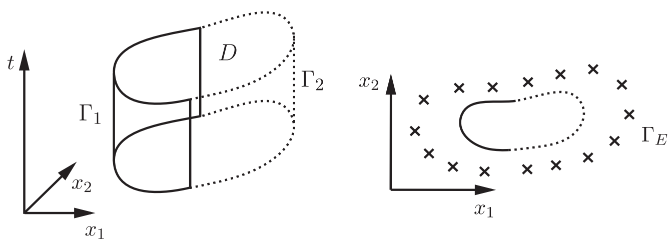

Fig. 2. Representation of a solution domain and the location of Cauchy data (—) on $\Gamma_T^{(1)}$, unknown boundary data ($\cdots$) on $\Gamma_T^{(2)}$, and the source points ($\times$) on Γ_E.

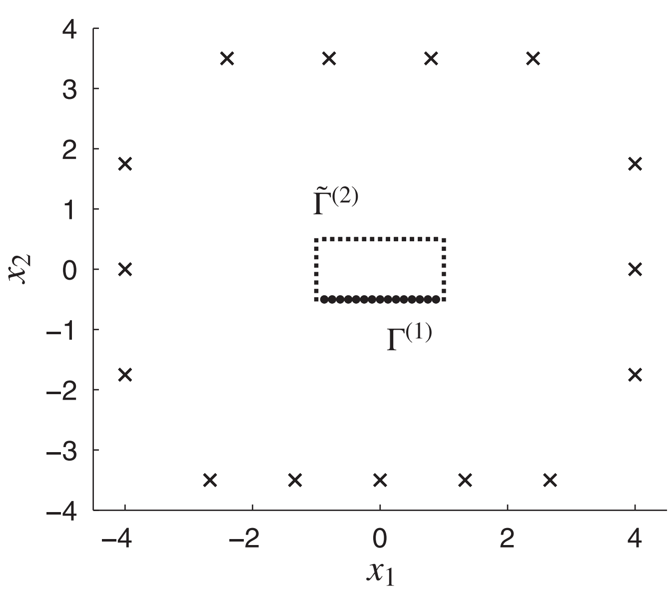

Fig. 3. Representation of the rectangular domain, Cauchy data points (•) on $\Gamma^{(1)}$, the unknown boundary data ($\cdots$) on $\Gamma^{(2)}$, and the source points ($\times$).

We wish to construct an approximation for the solution of the heat equation u, endowed with Dirichlet and Neumann conditions on the boundary $\Gamma_T^{(1)}$, i.e. to find an approximation to

$$\frac{\partial u}{\partial t}(\mathbf{x},t) = \Delta u(\mathbf{x},t), \quad (\mathbf{x},t) \in D_T, \tag{1}$$

$$u(\mathbf{x},t) = g_1(\mathbf{x},t), \quad (\mathbf{x},t) \in \Gamma_T^{(1)}, \tag{2}$$

$$\frac{\partial u}{\partial \boldsymbol{\nu}}(\mathbf{x},t) = g_2(\mathbf{x},t), \quad (\mathbf{x},t) \in \Gamma_T^{(1)}, \tag{3}$$

where $\boldsymbol{\nu}$ is the outward unit normal to the boundary Γ, and $\partial u / \partial \boldsymbol{\nu} = \nabla u \cdot \boldsymbol{\nu}$, and g_1 and g_2 are sufficiently smooth functions. We refer to the Dirichlet data in (2) as the temperature, and the Neumann data in (3) as the outward heat flux. Note that no initial condition is prescribed. The uniqueness of a solution is still guaranteed as is well known and explained in the sequel. Although it is known that initial data is not required for Cauchy problems for the heat equation, almost all numerical examples in the literature do have initial data imposed. It is noteworthy that with the MFS it is easy to work with or without this data.

For smooth Cauchy data given on a non-characteristic smooth curve (or surface in three-dimensions) uniqueness of a solution to the heat equation with this data is a consequence of Holmgren's theorem [13]. More general results, for example for non-characteristic Cauchy problems and also for the closely connected problem of unique continuation, have been presented in various function classes and solution domains. We do not aim to give an overview of these results and refer instead to two more recent papers which contain overviews of some of these [29,31]. In our situation with the smoothness imposed on the data and on the solution domain we can thus be certain that there can be at most one solution to the Cauchy problem. We shall assume that data is such that there exists a solution. Note though that this solution will not depend continuously on the data due to the ill-posedness of the Cauchy problem.

3. The MFS for the 2D heat conduction Cauchy problem

In this section, we construct an approximate solution to (1)–(3) using the MFS. To do this we need to make use of the fundamental solution of the two-dimensional heat equation (1), given by

$$F(\mathbf{x},t;\mathbf{y},\tau) = \frac{H(t-\tau)}{4\pi(t-\tau)} \exp\left(-\frac{|\mathbf{x}-\mathbf{y}|^2}{4(t-\tau)}\right), \tag{4}$$

having partial derivative with respect to x_1 and x_2, given by

$$\frac{\partial F}{\partial x_j}(\mathbf{x},t;\mathbf{y},\tau) = -\frac{H(t-\tau)}{8\pi(t-\tau)^2}(x_j - y_j) \exp\left(-\frac{|\mathbf{x}-\mathbf{y}|^2}{4(t-\tau)}\right), \quad j=1,2, \tag{5}$$

where H is the Heaviside function.

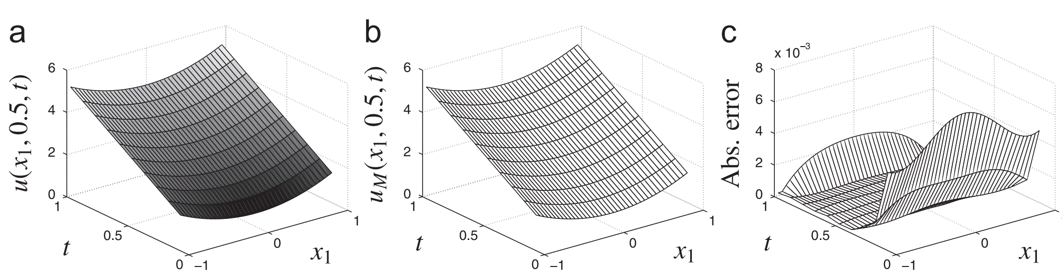

Fig. 4. (a) The temperature, (b) the MFS approximation and (c) the absolute error for $x_2 = 0.5$, $(x_1,t) = (-1,1) \times (0,1]$, obtained with $\lambda = 10^{-14}$, no noise $p_1 = p_2 = 0$, when Cauchy data is given along $x_2 = -0.5$, $(x_1,t) = (-1,1) \times (0,1]$, for Example 1.