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Corner and crack singularity of different boundary conditions for linear elastostatics and their numerical solutions

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ABSTRACT

This paper is devoted to study an important subject that the displacement and the free traction boundary conditions are assigned on the different edges of corners. Such different boundary conditions can often be found in engineering applications. Techniques for handling linear elastostatics with different boundary conditions are more intriguing and advanced, because the particular solutions are involved in $O(r^{\nu_k})$ with complex power ν_k, and because the complex coefficients are also needed. Two new model problems of corner singularity are designed for $\Theta=\pi$ and $\Theta=2\pi$, their highly accurate solutions and intensity factors can be achieved by the collocation Trefftz method. An advanced and comprehensive analysis of corner singularity of linear elastostatics of different boundary conditions is established in this paper, and the singular solutions can be used in numerical computation, thus to greatly enhance the Trefftz method, including the collocation Trefftz method, the hybrid Trefftz method, the combined method, etc.

1. Introduction

The singular solutions at corners and the fundamental solutions are important in both theory and computation for linear elastostatics. Our recent efforts are made to seek the particular solutions of corner and crack singularity of linear elastostatics, to design new models of different kinds of corner singularity, and to find their numerical solutions. In [1–3], the analysis is explored for singularity properties and particular solutions of linear elastostatics, and the singular solutions have been found for corners with three uniform boundary conditions: (1) the displacement condition, (2) the free traction condition, and (3) their mixed boundary conditions. This paper is an advanced study to explore different boundary conditions on different edges of corners. Compared with previous works, the techniques in this paper are more intriguing and challenging, because the particular solutions are involved in $O(r^{\nu_k})$ with complex power ν_k, and because the complex coefficients are also needed. The complex power ν_k can either be found by solving two nonlinear equations for general Θ or they can be expressed explicitly for real and imaginary parts on some special $\Theta=\ell\pi$, $\ell=1,2$, given when $\lambda=t\mu$ for any $t>0$, where λ and μ are the Lamé constants. Two new model problems of corner singularity are designed for $\Theta=\pi$ and $\Theta=2\pi$, their highly accurate solutions and intensity factors can be achieved by the collocation Trefftz method. To our best knowledge, this is the first time to provide the particular solutions near the corner with different boundary conditions on different edges in linear elastostatics, and to report their numerical computation. Since different boundary conditions often occur in engineering (e.g., the balcony of buildings with the L-shaped elastics structure, where one edge is free in the air, and the other edge is fixed on the wall), the analysis, computation and numerical results in this paper are significant for both theory and computation of linear elastostatics.

Here, let us mention some important works on this subject. The singular properties for traction conditions in Williams [4,5] are obtained by using a similarity to biharmonic equations, and some discussions and applications are followed by Lin and Tong [5], Jirousek and Venkstesh [6], Jirousek and Wroblewski [7] and Qin [8,2]. In [32,33], complex fundamental solutions and complex numerical methods are developed for plane elasticity problems. In this paper, we derive directly the particular solutions of linear elastostatics at corners. Once the singularity of corner solutions is known, the reduced convergence rates can be identified for finite element method (FEM), finite difference method (FDM) and finite volume method (FVM). Some improved techniques, such as the combined Trefftz method in [2,9], can be solicited to recover the optimal convergence rates. More importantly, by the results of this paper and [1–3], we may develop a number of new and renovated numerical methods for linear elastostatics, such as the combined method in [9–12], and the Trefftz methods [13–15], which include the boundary approximate method [9,16], the

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collocation Trefftz method [17–20], the hybrid Trefftz method (see [8,21–24]), the boundary collocation techniques [25,26], and elastoplastic stress stated in a cracked structure [27,28]. Results of these papers have established a deeper, comprehensive and systematic analysis of corner singularity of linear elastostatics. The particular solutions of crack singularity derived in this paper and [1–3] can be viewed as special purpose Trefftz functions, and other kinds of those functions for linear elastostatics are discussed in the special Trefftz method [37–42].

This paper is organized as follows. In Section 2, a basic description for linear elastostatics problems in 2D is introduced, and general particular solutions are provided. In Section 3, for displacement and traction boundary conditions at different corner edges, the powers ν_k of $O(r^{\nu_k})$ in the particular solutions are derived, and in Section 4, the explicit particular solutions are explored for the corner with the interior angles $\Theta=\pi,2\pi$. Two new models of crack singularity with different boundary conditions are designed for computation, one of them mimics Motz's model [29]. In Section 5, the collocation Trefftz method (CTM) in [9,18] is used to seek their solutions, and numerical results are reported, with the highly accurate solutions and intensity factors. In the last section, a few concluding remarks are addressed.

2. Linear elastostatics problems in 2D

Consider the linear elastostatics problem in 2D. Denote the displacement vector,

$$\vec{w}=\mathbf{w}=\{w_1(\mathbf{x}),w_2(\mathbf{x})\}^T=\{u(x,y),v(x,y)\}^T, \tag{2.1}$$

where $\vec{x}=\mathbf{x}=(x_1,x_2)=(x,y)$. The linear strain tensor is given by

$$\epsilon_{ij}(\mathbf{x})=\frac{1}{2}\left[\frac{\partial w_i(\mathbf{x})}{\partial x_j}+\frac{\partial w_j(\mathbf{x})}{\partial x_i}\right],\quad 1\le i,j\le 2. \tag{2.2}$$

Let σ_{ij} $(1\le i,j\le 2)$ denote the stress tensor at $\mathbf{x}$. For an isotropic Hookean solid, there exist the stress–strain relations

$$\sigma_{ij}=\lambda(\nabla\cdot\vec{w})\delta_{ij}+2\mu\epsilon_{ij},\quad 1\le i,j\le 2, \tag{2.3}$$

where "$\nabla\cdot$" is the divergence operator, δ_{ij} are the Kronecker delta, and $\lambda(>0)$ and $\mu(>0)$ are the Lamé constants. When there exists a body force $\vec{f}$, we obtain the non-homogeneous equation (see [30,31]),

$$\mu\Delta\vec{w}+(\lambda+\mu)\nabla(\nabla\cdot\vec{w})+\vec{f}=0\quad\text{in } S. \tag{2.4}$$

When $\vec{f}\equiv\vec{0}$, we have the Cauchy–Navier equation of linear elastostatics for isotropic body:

$$\Delta\vec{w}+\frac{1}{1-2v}\nabla(\nabla\cdot\vec{w})=0\quad\text{in } S, \tag{2.5}$$

where the Poisson ratio

$$v=\frac{\lambda}{2(\lambda+\mu)},\quad 0<v<\frac{1}{2}. \tag{2.6}$$

Denote the constant $\kappa=1/4(1-v)$. For the plane strain problem the constant

$$D=\frac{\lambda+\mu}{\lambda+3\mu}=\frac{1}{3-4v}=\frac{\kappa}{1-\kappa}, \tag{2.7}$$

and for the plane stress problem,

$$D=\frac{1+\hat{v}}{3-\hat{v}},\quad v=\frac{\hat{v}}{1+\hat{v}}. \tag{2.8}$$

The traction on ∂S is denoted by

$$\vec{\tau}(\vec{w})(\mathbf{x})=(\tau_x(u,v),\tau_y(u,v))^T, \tag{2.9}$$

where the stress components are given by

$$\tau_x(u,v)=\sigma_x\cos(n,x)+\sigma_{xy}\cos(n,y), \tag{2.10}$$

$$\tau_y(u,v)=\sigma_{xy}\cos(n,x)+\sigma_y\cos(n,y). \tag{2.11}$$

For the exterior normal $\vec{n}$ of the edge boundary, we have

$$\cos(n,x)=\cos\left(\frac{\pi}{2}+\theta\right)=-\sin\theta,\quad \cos(n,y)=\cos\theta. \tag{2.12}$$

Hence we have from (2.10) and (2.11)

$$\tau_x(u,v)=-\sigma_x\sin\theta+\sigma_{xy}\cos\theta,\quad \tau_y(u,v)=-\sigma_{xy}\sin\theta+\sigma_y\cos\theta, \tag{2.13}$$

where the stress

$$\sigma_x=(\lambda+2\mu)u_x+\lambda v_y,\quad \sigma_y=\lambda u_x+(\lambda+2\mu)v_y,\quad \sigma_{xy}=\mu(u_y+v_x). \tag{2.14}$$

In Jirousek and Wroblewski [7], Jirousek and Venkstesh [6], Qin [8], and [1–3], for the Cauchy–Navier equation (2.5), the particular solutions can be written as follows:

$$\begin{aligned}u_L=\sum_{k=1}^{L} r^{\nu_k}\{&a_k[-\sin\nu_k\theta+D\nu_k\sin(\nu_k-2)\theta]\\&+b_k[\cos\nu_k\theta-D\nu_k\cos(\nu_k-2)\theta]\\&+c_k\sin\nu_k\theta-d_k\cos\nu_k\theta\}+d_0,\end{aligned} \tag{2.15}$$

$$\begin{aligned}v_L=\sum_{k=1}^{L} r^{\nu_k}\{&a_k[\cos\nu_k\theta+D\nu_k\cos(\nu_k-2)\theta]\\&+b_k[\sin\nu_k\theta+D\nu_k\sin(\nu_k-2)\theta]\\&+c_k\cos\nu_k\theta+d_k\sin\nu_k\theta\}+c_0,\end{aligned} \tag{2.16}$$

where ν_k are complex, and a_k,b_k,c_k,d_k are complex coefficients.

3. Singularity near corners with different boundary conditions

The corner singularity of elasticity plane was first discussed in Williams [4] and Lin and Tong [5], and then in Jirousek and Wroblewski [7], Jirousek and Venkstesh [6] and Qin [8]. Note that Eqs. (2.15) and (2.16) derived directly from the Cauchy–Navier equation [3] are coincident with [6,8, p. 82].

We will derive the particular solutions for corners in this section, with the displacement condition on one edge, and the free traction boundary conditions on the other edge. Choose the sectorial domain $S=\{(r,\theta)|(0\le r<R,0\le\theta<\Theta)\}$, where $\Theta\in(0,2\pi]$, to satisfy

$$u=v=0,\quad\text{on }\theta=0, \tag{3.1}$$

$$\tau_x=\tau_y=0,\quad\text{on }\theta=\Theta. \tag{3.2}$$

First, consider the displacement condition (3.1). From (2.15) and (2.16) there exist the coefficients relations:

$$c_0=d_0=0,\quad d_k=b_k(1-D\nu_k),\quad c_k=-a_k(1+D\nu_k). \tag{3.3}$$

Then Eqs. (2.15) and (2.16) lead to

$$\begin{aligned}u_L=\sum_{k=1}^{L} r^{\nu_k}\{&a_k[-2\sin\nu_k\theta+D\nu_k(\sin(\nu_k-2)\theta-\sin\nu_k\theta)]\\&+b_kD\nu_k[\cos\nu_k\theta-\cos(\nu_k-2)\theta]\},\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}v_L=\sum_{k=1}^{L} r^{\nu_k}\{&a_kD\nu_k[\cos(\nu_k-2)\theta-\cos\nu_k\theta]\\&+b_k[2\sin\nu_k\theta+D\nu_k(\sin(\nu_k-2)\theta-\sin\nu_k\theta)]\}.\end{aligned} \tag{3.5}$$

3.1. The Powers ν_k in r^{ν_k}

We will use the basic approaches in [1–3], to seek the particular solutions of (3.4) and (3.5) satisfying the traction conditions (3.2). The