Journal of the Korean Statistical Society 46 (2017) 413–425

Contents lists available at ScienceDirect

Journal of the Korean Statistical Society

journal homepage: www.elsevier.com/locate/jkss

Optimal Berry–Esseen bound for an estimator of parameter in the Ornstein–Uhlenbeck process ☆

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ARTICLE INFO

Article history:
Received 21 September 2016
Accepted 16 January 2017
Available online 31 January 2017

AMS 2000 subject classifications:
primary 60H07
secondary 60F25

Keywords:
Malliavin calculus
Ornstein–Uhlenbeck process
Berry–Esseen bound
Maximum likelihood estimator
Multiple stochastic integral
Edgeworth expansion

ABSTRACT

This paper is concerned with the study of the rate of central limit theorem for the maximum likelihood estimator $\hat{\theta}_T$ of the unknown parameter $\theta > 0$, based on the observation $X = \{X_t, 0 \leq t \leq T\}$, occurring in the drift coefficient of an Ornstein–Uhlenbeck process $dX_t = -\theta X_t dt + dW_t, X_0 = 0$ for $0 \leq t \leq T$, where $\{W_t, t \geq 0\}$ is a standard Brownian motion. The tool we use is an *Edgeworth expansion* with an explicitly expressed remainder. We prove that upper and lower bounds, obtained by controlling the remainder term, give an optimal rate $\frac{1}{\sqrt{T}}$ in *Kolmogorov distance* for normal approximation of $\hat{\theta}_T$.

1. Introduction

In this paper, we find an optimal rate of convergence of the distribution of the maximum likelihood estimator (MLE) of the unknown parameter $\theta \in \Theta \subseteq \mathbb{R}^+$ based on the observation $X = \{X_t, 0 \leq t \leq T\}$ given by

$$dX_t = -\theta X_t dt + dW_t, \qquad X_0 = 0, \quad 0 \leq t \leq T, \tag{1}$$

where $\{W_t, t \geq 0\}$ is a standard Brownian motion. When the process $\{X_t, 0 \leq t \leq T\}$ can be observed, the MLE $\hat{\theta}_T$ is given by

$$\sqrt{\frac{T}{2\theta}}(\hat{\theta}_T - \theta) = \frac{-\sqrt{\frac{2\theta}{T}}S_T}{\frac{2\theta}{T}\langle S\rangle_T}, \tag{2}$$

where

$$S_T = \int_0^T X_t dW_t \quad \text{and} \quad \langle S\rangle_T = \int_0^T X_t^2 dt.$$

☆ This research was supported by Hallym University Research Fund, 2015 (HRF-201509-009), and by Basic Science Research Program through the National Research Foundation of Korea (NRF-2013R1A1A2008478).

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http://dx.doi.org/10.1016/j.jkss.2017.01.002
1226-3192/

Here we just employ the label $\langle S\rangle_T$ of the *quadratic variation* as the notation. It is well known that $\hat{\theta}_T$ is strongly consistent and $\sqrt{T/2\theta}(\hat{\theta}_T-\theta)$ converges to Gaussian random variable with the mean zero and unit variance as T tends to infinity (see Basawa & Prakasa Rao, 1980).

For the Berry–Esseen bound of the MLE $\hat{\theta}_T$, Mishra and Prakasa Rao in Mishra and Prakasa Rao (1985) obtained the rate $O(T^{-1/5})$ by using the technique of Michel and Pfanzagl (1971). In Bose (1985), author decomposed the numerator in (2) into two parts by using Itó formula, and obtained the rate $O(T^{-1/2}\log T)$. In Bishwal and Bose (1995), authors improve the rate to $O(T^{-1/2}\sqrt{\log T})$ by using a characteristic function for normal approximation of the numerator and moment generating function for the convergence of denominator. Afterwards, Bishwal (2000) improved the Berry–Esseen bound for $\hat{\theta}_T$ to $O(T^{-1/2})$ through the squeezing techniques of Pfanzagl (1971) developed for the minimum contrast estimator in Pfanzagl (1971) (for more information, see Bishwal, 2008).

The aim of the present work is to show that the upper bound $T^{-1/2}$ obtained by Bishwal (2000) is sharp by finding a lower bound with the same speed, that is an optimal Berry–Esseen bound for $\hat{\theta}_T$. As a tool for this, we use a one-term *Edgeworth expansion* from Kim and Park (submitted for publication). By using this method, we also find the upper bound $T^{-1/2}$ obtained by Bishwal (2000). We stress that our technique is more straightforward than the squeezing techniques used in the paper (Bishwal, 2000). Moreover, our technique is widely used for an optimal rate for parameter estimation of Gaussian processes.

We simply state the method of a one-term *Edgeworth expansion*. Let $\{F_n, n\geq 1\}$ be a sequence of random variables of functional of infinite-dimensional Gaussian fields associated with an isonormal Gaussian process defined on a probability space $(\Omega,\mathfrak{F},\mathbb{P})$. Authors (Kim & Park, submitted for publication), by combining Malliavin calculus and repeated applications of Stein's equations, find a one-term *Edgeworth expansion* with an explicit expression of the remainder $R_n(z)$:

$$\mathbb{P}(F_n\leq z)-\mathbb{P}(Z\leq z)=-\frac{1}{3!}H_2(z)\phi(z)\kappa_3(F_n)+R_n(z), \tag{3}$$

where $H_2(z)$ denotes the second Hermite polynomial, $\kappa_3(F_n)$ denotes the *the third cumulant* of F_n, Z is a standard Gaussian random variable and

$$\phi(z)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}.$$

For our work, we use this Edgeworth expansion (3) of the distribution function $\mathbb{P}\left(\sqrt{\frac{T}{2\theta}}(\hat{\theta}_T-\theta)\leq z\right)$. By controlling the remainder after required terms to obtain an upper (or lower) bound in the *Kolmogorov distance*, we obtain an optimal Berry–Esseen bound of a sequence $\left\{\sqrt{\frac{T}{2\theta}}(\hat{\theta}_T-\theta)\right\}$. We say that the bound $\varphi(T)$ is optimal for the sequence $\{F_T, T\geq 0\}$ with respect to the distance d if there exist constants $0<c<C<\infty$ (not depending on T) such that, for sufficiently large T,

$$c\leq\frac{d(F_T,N)}{\varphi(T)}\leq C. \tag{4}$$

In this paper, we focus on the normal approximation of random variables with respect to the *Kolmogorov distance* defined by

$$d(X,Y)=\sup_{z\in\mathbb{R}}|\mathbb{P}(X\leq z)-\mathbb{P}(Y\leq z)|.$$

The rest of the paper is organized as follows. Section 2 reviews some basic notations and results of Gaussian analysis and Malliavin calculus. In Section 3, we prove that the rate $T^{-1/2}$ is an optimal rate of CLT for the MLE $\hat{\theta}_T$.

2. Preliminaries

In this section, we recall some basic facts about Malliavin calculus for Gaussian processes. The reader is referred to Nourdin and Peccati (2012) and Nualart (2006) for a more detailed explanation. Suppose that $\mathfrak{H}$ is a real separable Hilbert space with scalar product denoted by $\langle\cdot,\cdot\rangle_{\mathfrak{H}}$. Let $B=\{B(h), h\in\mathfrak{H}\}$ be an isonormal Gaussian process, that is a centered Gaussian family of random variables such that $\mathbb{E}[B(h)B(g)]=\langle h,g\rangle_{\mathfrak{H}}$. For every $n\geq 1$, let $\mathbb{H}_n$ be the nth Wiener chaos of B, that is the closed linear subspace of $\mathbb{L}^2(\Omega)$ generated by $\{H_n(B(h)) : h\in\mathfrak{H}, \|h\|_{\mathfrak{H}}=1\}$, where H_n is the nth Hermite polynomial. We define a linear isometric mapping $I_n:\mathfrak{H}^{\odot n}\to\mathbb{H}_n$ by $I_n(h^{\otimes n})=n!H_n(B(h))$, where $\mathfrak{H}^{\odot n}$ is the symmetric tensor product. It is well known that any square integrable random variable $F\in L^2(\Omega,\mathcal{G},\mathbb{P})$ ($\mathcal{G}$ denotes the σ-field generated by B) can be expanded into a series of multiple stochastic integrals:

$$F=\sum_{k=0}^{\infty}I_k(f_k),$$

where $f_0=\mathbb{E}[F]$, the series converges in L^2, and the functions $f_k\in\mathfrak{H}^{\odot k}$ are uniquely determined by F.