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Time-dependent and stationary analyses of two-sided reflected Markov-modulated Brownian motion with bilateral ph-type jumps

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ABSTRACT

A Markov-modulated Brownian motion with bilateral ph-type jumps, referred to as MMBM, is a generalization of the Lévy process. In this paper, we study the time-dependent behavior of the two-sided reflected MMBM (TR-MMBM) with boundaries 0 and $\beta > 0$. In contrast to previous research on the subject, we propose a different approach based on the observation that the TR-MMBM can be realized as the limit of a sequence of two-sided reflected Markov-modulated fluid flows with bilateral ph-type jumps (TR-MMFF), which are MMBMs without a Brownian component. Therefore, the TR-MMBM can be analyzed via methods for the TR-MMFFs, through limiting arguments based on the weak convergence and continuous mapping theorems. Along these lines, we first analyze time-dependent behaviors of the sequence of TR-MMFFs using a new methodology that adopts the so-called completed graph and also using Markov renewal and skip-free level crossing arguments. Then, relying on the appropriate stochastic limit arguments, we finally present the Laplace transform of the time-dependent distribution of the TR-MMBM with respect to time. In addition, we show that the stationary distribution of the TR-MMBM can be obtained directly from the Laplace transform.

1. Introduction

The Markov-modulated Brownian motion with bilateral ph-type jumps, referred to as MMBM, is a bivariate Markov process $(X, J) = \{(X(t), J(t))\}$ such that: (i) J is an irreducible Markov process with infinitesimal generator Q and finite state space S; and (ii) for an interval where J is in $i \in S$, X evolves like Lévy process with a Lévy exponent

$$\kappa^{(i)}(s) = s\mu_i + s^2\sigma_i^2/2 + \int_{-\infty}^{\infty} [e^{sx} - 1 - sx\chi(|x| \le 1)]\nu_i(dx),$$

where $\chi(\cdot)$ denotes the indicator function; (iii) a transition from i to $j \neq i$ has probability p_{ij} of giving rise to a jump of X at the same time, with bilateral ph-type distribution B_{ij} of the form

$$B_{ij}(dx) = q_{ij}^{+}\chi(x > 0)\boldsymbol{\beta}_{ij}^{+}e^{T_{ij}^{+}x}(-T_{ij}^{+})\mathbf{1}dx + q_{ij}^{-}\chi(x < 0)\boldsymbol{\beta}_{ij}^{-}e^{-T_{ij}^{-}x}(-T_{ij}^{-})\mathbf{1}dx, \quad (1)$$

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where $q_{ij}^{\pm} \geq 0$ with $q_{ij}^{+} + q_{ij}^{-} = 1$, and $(\boldsymbol{\beta}_{ij}^{\pm}, T_{ij}^{\pm})$ are representations of phase-type distributions; and (iv) Lévy measures ν_i, inducing a compound Poisson process with bilateral ph-type jumps, have the form

$$\nu_i(dx) = \lambda_i^{+} \chi(x > 0) \boldsymbol{\beta}_{ii}^{+} e^{T_{ii}^{+} x} (-T_{ii}^{+}) \mathbf{1} dx + \lambda_i^{-} \chi(x < 0) \boldsymbol{\beta}_{ii}^{-} e^{-T_{ii}^{-} x} (-T_{ii}^{-}) \mathbf{1} dx, \tag{2}$$

where $\lambda_i^{\pm} \geq 0$, and $(\boldsymbol{\beta}_{ii}^{\pm}, T_{ii}^{\pm})$ are representations of phase-type distributions. Note that $\mathbf{1}$ is used to denote a vector of appropriate dimension with all entries being 1. For further details on the MMBM, we refer to Chapter XI of Asmussen (2003) and Breuer (2010).

The bilateral ph-type distribution, introduced by Ahn and Ramaswami (2005), can be represented as a mixture of two independent ph-type distributions as in (1) and (2), and is occasionally referred to as the double ph-type distribution (Jiang and Pistorius, 2008). We note that the class of bilateral ph-type distributions is dense in the space of distribution functions defined on the whole real line.

The two-sided reflected MMBM, referred to as TR-MMBM, is the image of the MMBM under the so-called two-sided reflection map with boundaries 0 and $\beta > 0$. This reflection map is a continuous mapping on $D[0, \infty)$, the set of all right-continuous functions with left limits defined on $[0, \infty)$, equipped with the uniform topology to be introduced later. For further details on the form of the two-sided reflection map and its continuity on $D[0, \infty)$, we refer to Kruck et al. (2007).

With reference to the latest research on the time-dependent and stationary distributions of the TR-MMBM, Ivanovs (2010) considered the TR-MMBM without a jump, and presented results on the time-dependent behavior of the model at inverse local times only when the initial level of the model is at boundaries 0 and β. He also provided simple probabilistic arguments for deriving the existing stationary result for the TR-MMBM without a jump. The same stationary result was derived by Ahn (2015) through a limit of the stationary distribution of the two-sided reflected Markov-modulated fluid flow without a jump, which weakly converges to the TR-MMBM without a jump. We note that the Markov-modulated fluid flow with bilateral ph-type jumps, referred to as MMFF, is a particular case of the MMBM without a Brownian component, and the two-sided reflected MMFF, referred to as TR-MMFF, is the image of MMFF under the two-sided reflection map.

In this paper, we investigate the time-dependent behavior of the TR-MMBM. To this end, we first obtain the Laplace transform with respect to time of the time-dependent distribution of the TR-MMFF using (i) a new methodology adopting the so-called completed graph that facilitates the application of the skip-free level crossing argument, and (ii) the Markov renewal theories. Then, using appropriate limit arguments based on the weak convergence and continuous mapping theorems, we present the complete formulae of the Laplace transforms of the time-dependent distributions of the TR-MMBM. To our knowledge, this finding and the approaches for it are novel to the literature.

Furthermore, we consider the TR-MMBM with $\beta = \infty$ to illustrate that taking the limit of the Laplace transform of the time-dependent distribution of the TR-MMBM can directly yield the corresponding stationary distribution.

The rest of this paper is organized as follows. We provide preliminaries in Section 2, in which we describe the MMBM of our interest, the sequence of the scaled MMFFs that weakly converge to the MMBM, the completed graph, and the existing results on the first passage time of the MMBM. This section also introduces the weak convergence and duality arguments, and the notations to be used in our analyses. The time-dependent analyses of the TR-MMFF, weakly convergent TR-MMFF, and TR-MMBM are respectively provided in Sections 3, 4, and 5. In Section 6, we derive the stationary distribution of the TR-MMBM with $\beta = \infty$ from its time-dependent distribution.

2. Preliminaries

2.1. Description of the MMBM and the limiting sequence of MMFFs

In relation to MMBM (X, J), we assume that state space S is partitioned as $S = S_b \cup S_u \cup S_d \cup S_0$ such that $\sigma_i > 0$ for $i \in S_b$ and $\sigma_i = 0$ for $i \notin S_b$; further, $\mu_i > 0, < 0$, and $= 0$ for $i \in S_u, \in S_d$, and $\in S_0$, respectively. The diffusion vector is denoted by $\boldsymbol{\sigma} = \{\sigma_i, i \in S\}$, and the drift (linear rate) vector by $\boldsymbol{\mu} = \{\mu_i, i \in S\}$. The sub-vectors $\boldsymbol{\sigma}_k$ and $\boldsymbol{\mu}_k$ of $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$ are defined such that $\boldsymbol{\sigma}_k = (\sigma_i, i \in S_k)$ and $\boldsymbol{\mu}_k = (\mu_i, i \in S_k)$ for $k = b, u, d, 0$. In addition, S_{j+} and S_{j-} denote the sets of states (phases) that originate from positive and negative ph-type jumps, respectively. We let $\boldsymbol{\pi} = (\boldsymbol{\pi}_b \ \boldsymbol{\pi}_u \ \boldsymbol{\pi}_d \ \boldsymbol{\pi}_0)$ be the stationary probability vector of Q satisfying $\boldsymbol{\pi} Q = \mathbf{0}$ and $\boldsymbol{\pi}\mathbf{1} = 1$. Note that subscripts $u, d, 0, j^{+}, j^{-}$, and b denote upward, downward, zero change, positive jump, negative jump, and Brownian motion, respectively.

Coordinate processes X and J of MMBM (X, J) are referred to as the level and phase processes, respectively. From the definition of the MMBM, level process X can be represented as

$$X(t) = \mathscr{G}(t) + \int_0^t \sigma_{J(u)} dB(u) \quad \text{with } \mathscr{G}(t) = a + \int_0^t \mu_{J(u)} du + \int_0^t dM(t), \tag{3}$$

where (i) M is the jump process with the jump sizes being bilateral ph-type distributed and the jump epochs being either the transition epochs of J, or the arrival epochs of the compound Poisson processes whose intensity depends on the state of J, and (ii) $B = \{B(t), t \geq 0\}$ is a standard Brownian motion that is independent of J and M.

$(\tilde{X}, \tilde{J})$ denotes the embedded process of (X, J). This is obtained using an embedding technique, that is, by replacing the positive and negative jumps in (X, J) with linear stretches having slopes $+1$ and -1, respectively, in $\tilde{X}$, and by letting $\tilde{J}$