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The random matrix regime of Maronna's estimator for observations corrupted by elliptical noise

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ABSTRACT

We study the behavior of Maronna's robust scatter estimator $\hat{C}_N \in \mathbb{C}^{N\times N}$ built from a sequence of observations $y_1, \ldots, y_n$ lying in a K-dimensional signal subspace of the N-dimensional complex field corrupted by heavy tailed noise, i.e., $y_i = A_N s_i + x_i$, where $A_N \in \mathbb{C}^{N\times K}$ and x_i is drawn from an elliptical distribution. In particular, we prove under mild assumptions that the robust scatter matrix can be characterized by a random matrix $\hat{S}_N$ that follows a standard random model as the population dimension N, the number of observations n, and the rank of A_N grow to infinity at the same rate. Our results are of potential interest for statistical theory and signal processing.

1. Introduction

Estimation of covariance matrices is at the heart of multivariate statistical analysis. It is important for a broad range of applications such as financial data analysis, statistical signal processing, and wireless communication. A classical estimator of the covariance matrix is the sample covariance matrix, defined as $\sum_{i=1}^{n} y_i y_i^*/n$, in terms of a random sample $y_1, \ldots, y_n$ from some N-variate distribution, with * denoting the complex conjugate transpose operator. The popularity of this estimator stems from its low complexity and the fact that it is the maximum likelihood estimator when observations are Gaussian.

Performance analysis of methods using the sample covariance matrix has attracted considerable attention in the last decades. An important result from random matrix theory states that the sample covariance matrix is no longer consistent when both n and N tend to infinity at the same rate. The need for improved methods has become more acute as a result of the recent trend in using large arrays for various applications.

Based on the enhanced understanding of the behavior of the sample covariance matrix that has accrued in recent years, a new wave of detection methods [1,3,15] and subspace estimation techniques [9,14,16] has emerged. These methods were designed to be consistent under mild regularity conditions. As they fundamentally rely on the sample covariance matrix, however, these improved methods can sometimes perform poorly. This occurs, e.g., in the presence of atypical observations making the Gaussian assumption inadequate. To overcome this issue, robust M-estimators of the covariance (or scatter) matrix have been proposed in [11,13]. We focus here on Maronna's estimator of scatter. This estimator turns out to be the maximum likelihood estimator for various complex elliptical distributions, which makes its use more natural than the sample covariance matrix.

Research studies analyzing the behavior of robust scatter estimators in general, and Maronna's estimator in particular, have typically been carried out under the assumption that the number of observations is large compared to the dimension.

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Recently, investigations of the large-dimensional regime in which n and N grow large at the same rate have become possible, thanks to new advanced tools developed in [6–8]. One major finding of these studies is that Maronna's estimator is equivalent in the asymptotic regime to a classical random matrix. Based on this equivalence, we conclude that random quantities depending on robust scatter estimators keep the same behavior when the robust estimator is replaced by its random equivalent. In doing so, we obtain standard random quantities, the behavior of which can be studied using standard results from random matrix theory.

Recent studies have begun to adapt these tools to radar and array processing applications that use robust scatter estimators [4,5]. In order to cope with the random model in array signal processing applications, the work in [4] extends the result of [6] to the spiked information plus noise model, in which observations are assumed to be composed of a low-dimensional Gaussian component representing often the signal contribution plus a high-dimensional random component modeling, in general, the noise contribution and drawn from an elliptical distribution.

The focus of the present work is high rank data models which, although less popular, are of interest for many existing applications. In these models, the Gaussian and elliptical components are high-dimensional. Similarly to [4], we show that Maronna's estimator has the same behavior as a random matrix following a standard model. Our result can have direct bearing on several applications, including the following two, which are representative.

1.1. Radar applications

Space Time Adaptive Processing (STAP) refers to the set of adaptive filtering algorithms that aim to suppress interference and improve target detection of a radar system. It is well known that effective interference rejection and improvements of detection performance require accurate estimation of the noise covariance matrix. In most radar applications, the noise can be modeled as a sum of two additive disturbances: a Gaussian noise, due to electronics, and the so-called clutter, of impulsive nature, which represents the response of the environment to the signal; see [2]. The observations are thus of the form $y = A_N s + x$, where $A_N A_N^*$ represents the correlation of the Gaussian noise matrix and x stands for the impulsive clutter. Note that the Gaussian component lies in a high-dimensional space, in accordance with the model considered here. In particular, adaptive beam-forming and radar detection fundamentally rely on the estimation of the covariance matrix. Below is a brief description of these two applications.

(a) Adaptive beam-forming: Let z_i be an $N \times 1$ multi-dimensional observation received by N linear antenna arrays at time $i \in \{1, \ldots, n\}$, viz. $z_i = s_i + y_i$, where $y_i = A_N s_i + x_i$ denotes the sum of interference and impulsive noise contributions. The output of a narrowband beam-forming vector is given by $d_i = w^* y_i$. The weight vector w should be set so as to maximize the signal-to-interference-plus-noise (SINR) given by $\mathrm{SINR} = w^* R_s w / w^* C_N w$, where R_s and C_N denote the covariance matrices of s_i and y_i, respectively. It can be shown easily that the beam-forming vector that maximizes the SINR is given by the principal eigenvector of the matrix $C_N^{-1} R_s$. Since C_N is in general unknown, it should be estimated using n signal free observations $y_1, \ldots, y_n$.

(b) Target detection: Consider the problem of detecting a complex signal vector p corrupted by an additive noise, viz. $z = \alpha p + y$, where $z \in \mathbb{C}^{N\times 1}$ represents the vector received by an N-dimensional array, y stands for the noise clutter, and α is a complex scalar modeling an unknown attenuation due to channel propagation. The signal detection problem can be cast as a test of the hypotheses $\mathcal{H}_0 : z = y$ vs. $\mathcal{H}_1 : z = \alpha p + y$. The Generalized Likelihood Ratio (GLRT) principle results in the test statistic

$$T_N = \frac{|z^* C_N^{-1} p|}{\sqrt{z^* C_N^{-1} z}\sqrt{p^* C_N^{-1} p}},$$

where C_N is the covariance matrix of the noise-plus-clutter vector y. Since the covariance matrix C_N is unknown in practice, a popular approach consists in replacing it by an estimate that is built on signal-free i.i.d. observations $y_1, \ldots, y_n$ termed secondary data. Due to the presence of an impulsive clutter, it is more sensible to use robust scatter estimators for the estimation of the covariance matrix. Studying the behavior of the resulting statistic would require establishing a Central Limit Theorem of quadratic forms derived from the robust scatter estimator. While these questions are not addressed directly in the present paper, our work contributes to some extent to lay the mathematical ground for solving them.

1.2. Blind subspace channel estimation

Consider a narrowband MIMO system with n transmitting antennas and N receiving antennas. Let s be the $n \times 1$ transmitted symbol vector. Then the received signal vector can be written as $y = A_N s + x$, where A_N denotes the MIMO channel and x the noise vector, which is assumed to be of an impulsive nature [17]. From subspace detection theory, we know that under mild conditions, $G^* A_N = 0$, where G contains the noise-subspace eigenvectors of the covariance matrix of y. Since the latter is unknown, an estimate must be used instead. Given that the noise is impulsive, it is sensible to employ a robust estimator, the properties of which need to be studied in order to obtain a better understanding of blind subspace channel estimation techniques.