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Extremal attractors of Liouville copulas

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ABSTRACT

Liouville copulas introduced in McNeil and Nešlehová (2010) are asymmetric generalizations of the ubiquitous Archimedean copula class. They are the dependence structures of scale mixtures of Dirichlet distributions, also called Liouville distributions. In this paper, the limiting extreme-value attractors of Liouville copulas and of their survival counterparts are derived. The limiting max-stable models, termed here the scaled extremal Dirichlet, are new and encompass several existing classes of multivariate max-stable distributions, including the logistic, negative logistic and extremal Dirichlet. As shown herein, the stable tail dependence function and angular density of the scaled extremal Dirichlet model have a tractable form, which in turn leads to a simple de Haan representation. The latter is used to design efficient algorithms for unconditional simulation based on the work of Dombry et al. (2016) and to derive tractable formulas for maximum-likelihood inference. The scaled extremal Dirichlet model is illustrated on river flow data of the river Isar in southern Germany.

1. Introduction

Copula models play an important role in the analysis of multivariate data and find applications in many areas, including biostatistics, environmental sciences, finance, insurance, and risk management. The popularity of copulas is rooted in the decomposition of Sklar [39], which is at the heart of flexible statistical models and various measures, concepts and orderings of dependence between random variables. According to Sklar's result, the distribution function of any random vector $\boldsymbol{X} = (X_1, \ldots, X_d)$ with continuous univariate margins $F_1, \ldots, F_d$ satisfies, for any $x_1, \ldots, x_d \in \mathbb{R}$,

$$\Pr(X_1 \le x_1, \ldots, X_d \le x_d) = C\{F_1(x_1), \ldots, F_d(x_d)\},$$

for a unique copula C, i.e., a distribution function on $[0, 1]^d$ whose univariate margins are standard uniform. Alternatively, Sklar's decomposition also holds for survival functions, i.e., for any $x_1, \ldots, x_d \in \mathbb{R}$,

$$\Pr(X_1 > x_1, \ldots, X_d > x_d) = \hat{C}\{\bar{F}_1(x_1), \ldots, \bar{F}_d(x_d)\},$$

where $\bar{F}_1, \ldots, \bar{F}_d$ are the marginal survival functions and $\hat{C}$ is the survival copula of $\boldsymbol{X}$, related to the copula of $\boldsymbol{X}$ as follows. If $\boldsymbol{U}$ is a random vector distributed as the copula C of $\boldsymbol{X}$, $\hat{C}$ is the distribution function of $1 - \boldsymbol{U}$.

In risk management applications, the extremal behavior of copulas is of particular interest, as it describes the dependence between extreme events and consequently the value of risk measures at high levels. Our purpose is to study the extremal behavior of Liouville copulas. The latter are defined as the survival copulas of Liouville distributions [14,17,38], i.e., distributions

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of random vectors of the form $R\boldsymbol{D}_{\boldsymbol{\alpha}}$, where R is a strictly positive random variable independent of the Dirichlet random vector $\boldsymbol{D}_{\boldsymbol{\alpha}} = (D_1, \ldots, D_d)$ with parameter vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$. Liouville copulas were proposed by McNeil and Nešlehová [31] in order to extend the widely used class of Archimedean copulas and create dependence structures that are not necessarily exchangeable. The latter property means that for any $u_1, \ldots, u_d \in [0, 1]$ and any permutation π of the integers $1, \ldots, d$, $C(u_1, \ldots, u_d) = C(u_{\pi(1)}, \ldots, u_{\pi(d)})$. When $\boldsymbol{\alpha} = \mathbf{1}_d \equiv (1, \ldots, 1)$, $\boldsymbol{D}_{\boldsymbol{\alpha}} = \boldsymbol{D}_{\mathbf{1}_d}$ is uniformly distributed on the unit simplex

$$\mathbb{S}_d = \{\boldsymbol{x} \in [0, 1]^d : x_1 + \cdots + x_d = 1\}. \tag{1}$$

In this special case, one recovers Archimedean copulas. Indeed, according to [30], the latter are the survival copulas of random vectors $R\boldsymbol{D}_{\mathbf{1}_d}$, where R is a strictly positive random variable independent of $\boldsymbol{D}_{\mathbf{1}_d}$. When $\boldsymbol{\alpha} \neq \mathbf{1}_d$, the survival copula of $R\boldsymbol{D}_{\boldsymbol{\alpha}}$ is not Archimedean anymore. It is also no longer exchangeable, unless $\alpha_1 = \cdots = \alpha_d$.

In this article, we determine the extremal attractor of a Liouville copula and of its survival counterpart. As a by-product, we also obtain the lower and upper tail dependence coefficients of Liouville copulas that quantify the strength of dependence at extreme levels [25]. These results are complementary to [21], where the upper tail order functions of a Liouville copula and its density are derived when $\alpha_1 = \cdots = \alpha_d$, and to [20], where the extremal attractor of $R\boldsymbol{D}_{\boldsymbol{\alpha}}$ is derived when R is light-tailed. The extremal attractors of Liouville copulas are interesting in their own right. Because non-exchangeability of Liouville copulas carries over to their extremal limits, the latter can be used to model the dependence between extreme risks in the presence of causality relationships [15]. The limiting extreme-value models can be embedded in a single family, termed here the scaled extremal Dirichlet, whose members are new, non-exchangeable generalizations of the logistic, negative logistic, and Coles–Tawn extremal Dirichlet models given in [7]. We examine the scaled extremal Dirichlet model in detail and derive its de Haan spectral representation. The latter is simple and leads to feasible stochastic simulation algorithms and tractable formulas for likelihood-based inference.

The article is organized as follows. The extremal behavior of the univariate margins of Liouville distributions is first studied in Section 2. The extremal attractors of Liouville copulas and their survival counterparts are then derived in Section 3. When $\boldsymbol{\alpha}$ is integer-valued, the results of [27,31] lead to closed-form expressions for the limiting stable tail dependence functions, as shown in Section 4. Section 5 is devoted to a detailed study of the scaled extremal Dirichlet model. In Section 6, the de Haan representation is derived and used for stochastic simulation. Estimation is investigated in Section 7, where expressions for the censored likelihood and the gradient score are also given. An illustrative data analysis of river flow of the river Isar is presented in Section 8, and the paper is concluded by a discussion in Section 9. Lengthy proofs are relegated to the Appendices.

In what follows, vectors in $\mathbb{R}^d$ are denoted by boldface letters, $\boldsymbol{x} = (x_1, \ldots, x_d)$; $\mathbf{0}_d$ and $\mathbf{1}_d$ refer to the vectors $(0, \ldots 0)$ and $(1, \ldots, 1)$ in $\mathbb{R}^d$, respectively. Binary operations such as $\boldsymbol{x} + \boldsymbol{y}$ or $a \cdot \boldsymbol{x}$, $\boldsymbol{x}^a$ are understood as component-wise operations. $\|\cdot\|$ stands for the ℓ_1-norm, viz. $\|\boldsymbol{x}\| = |x_1| + \cdots + |x_d|$, $\perp\!\!\!\perp$ for statistical independence. For any $x, y \in \mathbb{R}$, let $x \wedge y = \min(x, y)$ and $x \vee y = \max(x, y)$. The Dirac delta function I_{ij} is 1 if $i = j$ and zero otherwise. Finally, $\mathbb{R}^d_+$ is the positive orthant $[0, \infty)^d$ and for any $x \in \mathbb{R}$, x_+ denotes the positive part of x, $\max(0, x)$.

2. Marginal extremal behavior

A Liouville random vector $\boldsymbol{X} = R\boldsymbol{D}_{\boldsymbol{\alpha}}$ is a scale mixture of a Dirichlet random vector $\boldsymbol{D}_{\boldsymbol{\alpha}} = (D_1, \ldots, D_d)$ with parameters $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d) > \mathbf{0}_d$. In what follows, R is referred to as the radial variable of $\boldsymbol{X}$ and $\bar{\alpha}$ denotes the sum of the Dirichlet parameters, viz. $\bar{\alpha} = \|\boldsymbol{\alpha}\| = \alpha_1 + \cdots + \alpha_d$. Recall that $\boldsymbol{D}_{\boldsymbol{\alpha}}$ has the same distribution as $\boldsymbol{Z}/\|\boldsymbol{Z}\|$, where $Z_1 \sim \mathcal{G}(\alpha_1, 1), \ldots, Z_d \sim \mathcal{G}(\alpha_d, 1)$ are independent Gamma variables with scaling parameter 1. The margins of $\boldsymbol{X}$ are thus scale mixtures of Beta distributions, i.e., for each $i \in \{1, \ldots, d\}$, $X_i = RD_i$ with $D_i \sim \mathcal{B}(\alpha_i, \bar{\alpha} - \alpha_i)$.

As a first step towards the extremal behavior of Liouville copulas, this section is devoted to the extreme-value properties of the univariate margins of the vectors $\boldsymbol{X}$ and $1/\boldsymbol{X}$, where $\boldsymbol{X}$ is a Liouville random vector with parameters $\boldsymbol{\alpha}$ and a strictly positive radial part R, i.e., such that $\Pr(R \leq 0) = 0$. To this end, recall that a univariate random variable X with distribution function F is in the maximum domain of attraction of a non-degenerate distribution F_0, denoted $F \in \mathcal{M}(F_0)$ or $X \in \mathcal{M}(F_0)$, if and only if there exist sequences of reals (a_n) and (b_n) with $a_n > 0$, such that, for any $x \in \mathbb{R}$,

$$\lim_{n \to \infty} F^n(a_n x + b_n) = F_0(x).$$

By the Fisher–Tippett Theorem, F_0 must be, up to location and scale, either the Fréchet (Φ_ρ), the Gumbel (Λ) or the Weibull distribution (Ψ_ρ) with parameter $\rho > 0$. Further recall that a measurable function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is called regularly varying with index $\rho \in (-\infty, \infty)$, denoted $f \in \mathcal{R}_\rho$, if for any $x > 0$, $f(tx)/f(t) \to x^\rho$ as $t \to \infty$. If $\rho = 0$, f is called slowly varying. For more details and conditions for $F \in \mathcal{M}(F_0)$, see, e.g., [12,35].

Because the univariate margins of $\boldsymbol{X}$ are scale mixtures of Beta distributions, their extremal behavior, detailed in Proposition 1, follows directly from Theorems 4.1, 4.4. and 4.5 in [19].

Proposition 1. *Let $\boldsymbol{X} = R\boldsymbol{D}_{\boldsymbol{\alpha}}$ be a Liouville random vector with parameters $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$ and a strictly positive radial variable R, i.e.,* $\Pr(R \leq 0) = 0$. *Then the following statements hold for any $\rho > 0$:*