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On optimal grouping and stochastic comparisons for heterogeneous items

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ABSTRACT

In this paper, we consider series and parallel systems composed of n independent items drawn from a population consisting of m different substocks/subpopulations. We show that for a series system, the optimal (maximal) reliability is achieved by drawing all items from one substock, whereas, for a parallel system, the optimal solution results in an independent drawing of all items from the whole mixed population. We use the theory of stochastic orders and majorization orders to prove these and more general results. We also discuss possible applications and extensions.

1. Introduction and preliminaries

Optimal grouping of components (items) in engineering systems (to achieve maximal reliability) is often case-specific. Therefore, it is important to have some general approaches for such problems. The following introductory example will help us to illustrate the problem discussed in this paper.

Consider a coherent system of independent and identically distributed (i.i.d.) items subjected to an external shock process(es); see [3,4]. Assume for simplicity that shocks constitute the only cause of item failure. For definiteness (but not necessarily), we can think of the extreme shock model when each shock "independently of everything else" results in any item's failure with probability p and is harmless with probability $q = 1 - p$ (specifically, $p = 1$ corresponds to the case when the first arriving shock "kills" an item). Assume now that we have the ability to group items in such a way that each group is exposed to its own shock process. All shock processes are independent and statistically identical. The latter, e.g., for the Poisson process of shocks, means that all rates of the processes are equal. Usually this grouping (or, alternatively, separation) is performed to increase the resilience of the whole system to shocks. However, obtaining an optimal grouping strategy for the case of a general coherent system of i.i.d. items is still an open problem due to its complexity. Nevertheless, the answer for the series and parallel systems is intuitively obvious and can be formally proved. Indeed, for series systems of i.i.d. components, the optimal grouping can be achieved when the whole system is exposed to one process of shocks. This is due to the positive correlation of lifetimes in this system resulting from the mutual process of shocks. Thus separation, if

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implemented, will only increase the probability that at least one of the components will not survive a shock, which is equal to the probability of a failure of the entire system. Similar considerations hold for parallel systems, where an optimal solution is a total separation, i.e., each item should be exposed to its own independent process of shocks.

Cha [1] considered a somewhat related problem in a specific context through direct comparisons of survival probabilities induced by the corresponding non-homogeneous Poisson processes. Similar problems have also been discussed in [2,6,8]. These authors have also emphasized the practical importance of optimal grouping in various applications including transportation optimization, job scheduling, investment risk minimization, optimal allocation strategies, etc. It should be noted that the current paper deals with a much more general setting. Moreover, the methodology developed here is based on the theory of stochastic orders and provides more general comparative results as well. For instance, along with conventional ordering of the corresponding survival functions (usual stochastic order), we also deal with the stronger hazard rate order and intensively employ the majorization order for the vectors of interest. All this shows the innovative nature of our stochastic approach to a variety of real-world problems. In what follows in this section, we present some basic notation and relevant definitions.

We use the following basic notation for the corresponding random variables. For an absolutely continuous random variable Y, we denote the probability density function (pdf) by f_Y, the cumulative distribution function (cdf) by F_Y, and the hazard (failure) rate function by r_Y, and the reversed hazard rate function by $\tilde{r}_Y$. The survival or reliability function of the random variable Y is written as $\bar{F}_Y = 1 - F_Y$. We denote the set of real numbers by $\mathbb{R}$ and the set of natural numbers by $\mathbb{N}$.

For convenience, we now define a few types of orderings that will be used throughout this paper. We start with the meaningful majorization order.

The majorization orders and the Schur-convexity are quite useful to establish various inequalities. In the literature, different kinds of majorization orders have been developed to study different types of problems in various areas of mathematics, statistics, economics, physics and so on. The following well known definition of the majorization order could be found in [7].

Definition 1. Let S^n denote an n-dimensional Euclidean space where $S \subseteq \mathbb{R}$. Further, let $\boldsymbol{x} = (x_1, \ldots, x_n) \in S^n$ and $\boldsymbol{y} = (y_1, \ldots, y_n) \in S^n$ be any two vectors, and $x_{(1)} \le \cdots \le x_{(n)}$ and $y_{(1)} \le \cdots \le y_{(n)}$ be the increasing arrangements of the components of $\boldsymbol{x}$ and $\boldsymbol{y}$, respectively. The vector $\boldsymbol{x}$ is said to majorize the vector $\boldsymbol{y}$ (written as $\boldsymbol{x} \overset{ma}{\succeq} \boldsymbol{y}$) if

$$\forall_{j \in \{1,\ldots,n-1\}} \quad \sum_{i=1}^{j} x_{(i)} \le \sum_{i=1}^{j} y_{(i)} \quad \text{and} \quad \sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}.$$

Definition 2. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called Schur-convex if $\boldsymbol{x} \overset{ma}{\succeq} \boldsymbol{y} \Rightarrow f(\boldsymbol{x}) \ge f(\boldsymbol{y})$.

Stochastic orderings are important tools to compare random variables. In the literature, many different types of stochastic orders have been developed to handle different types of problems; see [5,9]. For the sake of completeness, we provide the following definitions of stochastic orders that will be used throughout our paper. In what follows, increasing and decreasing refer to non-decreasing and non-increasing, respectively. Furthermore, $a \overset{\text{sgn}}{=} b$ means that a and b have the same sign. For convenience, we use bold symbols to represent vectors.

Definition 3. Let X and Y be two continuous nonnegative random variables with respective supports (ℓ_X, u_X) and (ℓ_Y, u_Y), where u_X and u_Y may be positive infinity, and $\ell_X > 0$ and $\ell_Y > 0$. Then X is said to be smaller than Y in

(i) the usual stochastic (st) order, denoted as $X \le_{st} Y$, if $\bar{F}_X(x) \le \bar{F}_Y(x)$ for all $x \in (0, \infty)$;
(ii) the hazard rate (hr) order, denoted as $X \le_{hr} Y$, if $\bar{F}_Y(x)/\bar{F}_X(x)$ is increasing in $x \in (0, \max(u_X, u_Y))$;
(iii) the reversed hazard rate (rhr) order, denoted as $X \le_{rhr} Y$, if $F_Y(x)/F_X(x)$ is increasing in $x \in (\min(\ell_X, \ell_Y), \infty)$.

The following diagram shows the chain of implications among the stochastic orders as discussed above.

$$\begin{array}{ccc} & X \le_{st} Y & \\ \nearrow & & \nwarrow \\ X \le_{hr} Y & & X \le_{rhr} Y. \end{array}$$

The rest of this paper is organized as follows. In Section 2, we describe our setting. In Section 3, various comparative results are presented. Finally, in Section 4, brief conclusions are given.

2. The setting

In this paper, we consider series and parallel systems composed of n independent items drawn from a mixed population consisting of m different substocks/subpopulations (i.e., large lots of items). We obtain meaningful comparison results for these simplest structures. We plan to address the case of general coherent systems of independent items in our future research. For convenience, we use the terms “subpopulation” and “substock” interchangeably. In practice, there are various