Journal of Multivariate Analysis 155 (2017) 52–71

Contents lists available at ScienceDirect

Journal of Multivariate Analysis

journal homepage: www.elsevier.com/locate/jmva

Properties of extremal dependence models built on bivariate max-linearity

Mónika Kereszturi *, Jonathan Tawn

STOR-i Centre for Doctoral Training, Lancaster University, UK

ARTICLE INFO

Article history:
Received 18 May 2016
Available online 9 December 2016

AMS subject classifications:
60G70
62G32
62E20
62G20

Keywords:
Bivariate extremes
Max-linear models
Extremal dependence
Asymptotic independence

ABSTRACT

Bivariate max-linear models provide a core building block for characterizing bivariate max-stable distributions. The limiting distribution of marginally normalized component-wise maxima of bivariate max-linear models can be dependent (asymptotically dependent) or independent (asymptotically independent). However, for modeling bivariate extremes they have weaknesses in that they are exactly max-stable with no penultimate form of convergence to asymptotic dependence, and asymptotic independence arises if and only if the bivariate max-linear model is independent. In this work we present more realistic structures for describing bivariate extremes. We show that these models are built on bivariate max-linearity but are much more general. In particular, we present models that are dependent but asymptotically independent and others that are asymptotically dependent but have penultimate forms. We characterize the limiting behavior of these models using two new different angular measures in a radial–angular representation that reveal more structure than existing measures.

1. Introduction

When modeling extremes of spatial environmental processes we often care about both local dependence and long-range dependence. For example, in an oceanographic application, we would be interested in the relationship between extreme significant wave heights at two locations that might be close by or located far apart. In particular, we want to know how likely it is that both locations are affected by the same storm and have high waves simultaneously; see, e.g., [8]. Since interest lies in the extremes, the standard measures of spatial dependence are not appropriate and alternative dependence measures and models should be used. Here we introduce a family of bivariate distributions, with simple multivariate extensions, that exhibits all the required features of short, medium and long range extremal dependence for spatial applications. This family is shown to capture all possible bivariate distributions with these properties. We propose novel bivariate characterizations of the extremal dependence structure that reveal structure of this family of distributions that standard measures of extremal dependence fail to identify.

First we identify the two core extremal dependence measures. Let X and Y be identically distributed random variables. Then, an intuitive measure of extremal dependence is the tail dependence measure χ, which is defined as the limiting probability that Y is extreme given that X is extreme,

$$\chi = \lim_{z \to z_F} \Pr(Y > z \mid X > z), \tag{1}$$

* Corresponding author.
E-mail address: m.kereszturi@lancaster.ac.uk (M. Kereszturi).

http://dx.doi.org/10.1016/j.jmva.2016.12.001

where z_F is the upper end point of the common marginal distribution. When $\chi > 0$, X and Y are said to be asymptotically dependent (AD) and the value of χ signifies the strength of asymptotic dependence. This means that X and Y can be extreme simultaneously. However, when the variables are asymptotically independent (AI), $\chi = 0$ and hence χ does not contain any information about the sub-asymptotic dependence structure. Coles et al. [1] argue that to give a more complete summary of extremal dependence a second measure is needed to describe the rate of convergence of $\Pr(Y > z \mid X > z)$ to 0. A useful tail dependence measure can be obtained from the Ledford and Tawn [12] joint tail dependence model, which states that

$$\Pr(X > z, Y > z) = \mathcal{L}\{1/\Pr(X > z)\}\{\Pr(X > z)\}^{2/(\bar{\chi}+1)}, \tag{2}$$

where $\mathcal{L}$ is a slowly varying function at infinity and $\bar{\chi} \in (-1, 1]$. The exponent $2/(\bar{\chi} + 1)$ determines the decay rate of the joint probability, with smaller $\bar{\chi}$ giving more rapid convergence of χ to 0. The pair ($\chi > 0$; $\bar{\chi} = 1$) signifies AD, for which the value of χ gives a measure of strength of dependence; and ($\chi = 0$; $\bar{\chi} < 1$) signifies AI, for which the value of $\bar{\chi}$ gives the strength of dependence.

Both the dependence measures χ and $\bar{\chi}$, in expressions (1) and (2), are invariant to the marginal distribution. Of course, using the concept of copulas, all dependence measures can be expressed independently of the marginal distributions. However, for some choices of marginal distributions extremal dependence structure properties are more simply expressed than for other marginal choices. For example, much of the traditional multivariate extreme value theory results are expressed for Fréchet marginals, as they lead to the cleanest expressions of results for component-wise maxima and multivariate regular variation [17]. This marginal choice is fine when the variables are AD, however for AI variables this selection leads to an identical limit form whatever the nature of the AI, i.e., whatever $\bar{\chi} < 1$. For AI variables, [7,9,20] all identify that non-degenerate limit distributions, under affine transformations, can be obtained using exponential margins/tails, whereas under their formulations the limits are degenerate for Fréchet margins. Furthermore, in exponential margins results for AD are also non-degenerate. The reason for this extra flexibility in exponential margins is that an affine transformation in that space is a complex non-linear transformation in Fréchet margins; see Section 2.2 of [15]. Therefore, we work in exponential margins to illustrate our novel extremal dependence characterizations and show that if Fréchet margins had been used, the structure we find would not have been apparent using affine transformations.

In the analysis of multivariate data, it is often difficult to make a choice between AD and AI; see, e.g., [3,11,18]. By having a model that has both AD and AI components, we can avoid having to make this key decision. Wadsworth and Tawn [19] combine a max-stable process with an inverted max-stable process to construct a hybrid spatial dependence model. This model can capture both the AD and AI dependence structure but it is restricted in its forms of AD and AI that can be modeled. Here we use the core structure of the Wadsworth and Tawn [19] model as a basis for exploring bivariate extreme value modeling in a new light. Specifically, we develop a distribution that contains both AD and AI components and has the flexibility to capture all dependence forms within very broad classes in each case.

We construct our model using the multivariate max-linear model [2] as the building block. This class of distributions is both mathematically elegant and the starting point for understanding the formulation of multivariate extremes [16]. In the bivariate case with Fréchet marginal variables X_F and Y_F, the max-linear model takes the following form:

$$X_F = \max_{i=1,\ldots,m}(\alpha_i Z_i), \qquad Y_F = \max_{i=1,\ldots,m}(\beta_i Z_i), \tag{3}$$

where $\alpha_i, \beta_i \in [0, 1]$ for all i, m can be finite or infinite, $\Sigma_{i=1}^m \alpha_i = 1$, $\Sigma_{i=1}^m \beta_i = 1$, and $Z_i \sim$ i.i.d. Fréchet, $i = 1, \ldots, m$, with distribution function $F_Z(z) = \exp(-1/z)$ for $z > 0$ and density denoted $f_Z(z)$. This model has joint distribution function

$$\Pr(X_F < x, Y_F < y) = \exp\left\{-\sum_{i=1}^{m}\max\left(\frac{\alpha_i}{x}, \frac{\beta_i}{y}\right)\right\}, \quad \text{for } x > 0,\ y > 0,$$

and it is straightforward to show that this satisfies max-stability, since for any $n > 0$, $x > 0$ and $y > 0$,

$$\Pr(X_F < nx, Y_F < ny)^n = \Pr(X_F < x, Y_F < y).$$

Fundamental to our approach is that Deheuvels [5] shows that every multivariate extreme value distribution for minima, with exponential marginals (i.e., with variables (X_F^{-1}, Y_F^{-1})), can be arbitrarily well approximated by a multivariate max-linear model. Fougères et al. [6] showed this property holds for (X_F, Y_F), as well as presenting a broader discussion on alternative representations of multivariate extreme value distributions.

Our paper introduces two bivariate distributions, with exponential margins, that are derived from the max-linear model (3) with Fréchet margins: these are the transformed max-linear model and the inverted max-linear model, denoted by (X_E, Y_E) and $(X_E^{(I)}, Y_E^{(I)})$ respectively. Specifically,

$$(X_E, Y_E) = (-\ln\{1 - \exp(-1/X_F)\}, -\ln\{1 - \exp(-1/Y_F)\}) \tag{4}$$

and

$$(X_E^{(I)}, Y_E^{(I)}) = (1/X_F, 1/Y_F). \tag{5}$$

Here X_E (Y_E) transforms X_F (Y_F) to the exponential margins through a monotone increasing mapping, with repeated use of the probability integral transform, whereas $X_E^{(I)}$ ($Y_E^{(I)}$) transforms X_F (Y_F) to the exponential margins through a