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Some asymptotic theory for Silverman's smoothed functional principal components in an abstract Hilbert space

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ABSTRACT

Unlike classical principal component analysis (PCA) for multivariate data, one needs to smooth or regularize when estimating functional principal components. Silverman's method for smoothed functional principal components has nice theoretical and practical properties. Some theoretical properties of Silverman's method were obtained using tools in the L^2 and the Sobolev spaces. This paper proposes an approach, in a general manner, to study the asymptotic properties of Silverman's method in an abstract Hilbert space. This is achieved by exploiting the perturbation results of the eigenvalues and the corresponding eigenvectors of a covariance operator. Consistency and asymptotic distributions of the estimators are derived under mild conditions. First we restrict our attention to the first smoothed functional principal component and then extend the same method for the first K smoothed functional principal components.

1. Introduction

There are many situations in applied sciences where collected data are originally functions, curves or surfaces rather than vectors. Functional Data Analysis (FDA) is a major branch of research in statistics which deals with data of a continuous nature. In the recent past, a wide scope of contributions related to FDA has been made. The book by Horváth and Kokoszka [18] discusses theoretical and practical aspects related to principal components analysis, time series, change point detection, and spatial statistics in FDA. The essential mathematical concepts and results relevant to the theoretical development of FDA are explored in Hsing and Eubank [19]. The recent reviews by Cuevas [8] and Goia and Vieu [15] survey the main ideas and current literature on regression, classification, bootstrap methods, dimension reduction, semi-parametric modeling, and nonparametric techniques in FDA.

Since functional data are infinite-dimensional, we need to extract important information in order to get a thorough understanding of the structure of the data. Functional principal component analysis (FPCA) provides a finite basis system to represent infinite-dimensional functional data with high accuracy. Many important features and the dominant modes of variations of data are captured by these functional principal components. Early work on FPCA can be found in Deville [10], Dauxois et al. [9], Besse and Ramsay [4], Ramsay and Dalzell [30], Castro et al. [7], and Rice and Silverman [33].

There is a rich collection of literature related to recent work on FPCA. Chapter 8 of Ramsay and Silverman [31] gives a comprehensive discussion of FPCA in the context of the basis representation of the functional data. The properties of FPCA are explained through stochastic expansion and related results in [16]. Kokoszka and Reimherr [24] establish the asymptotic normality of the sample principal components of functional time series data. There are different ways of obtaining estimates

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in FPCA. Using penalized spline regression, Yao and Lee [36] propose an iterative estimation method for performing FPCA. Ocaña et al. [27] establish a procedure to formulate an algorithm to compute estimates of FPCA under general settings. An approach for robust estimators of functional principal components is given in [3].

FPCA has been applied to many theoretical and practical problems. A direct application of FPCA is functional principal component regression; see, e.g., Cardot et al. [6]. In the context of longitudinal data analysis, a random function usually represents an individual or an item observed at a small number of random points. Hall et al. [17] discuss the application of FPCA to longitudinal data. There are many other practical applications of FPCA in a variety of fields such as the analysis of income density curves [23], spectroscopy data [37], and financial time series data [22].

Since non-smooth functional principal components are too rough for accurate interpretations and advanced analysis, we need to smooth or regularize when estimating them. Many approaches have been proposed to estimate smoothed functional principal components. In one approach, data are smoothed first and FPCA is performed on the smoothed data. Kernel smoothed FPCA is based on this approach, and the asymptotic properties of these principal components are discussed in [5]. Rice and Silverman [33] propose another approach to smoothed FPCA where the variance of principal components is penalized based on a roughness penalty. Rather than penalizing the variance, Silverman [35] incorporates the roughness penalty into the orthonormality constraint in performing smoothed FPCA. An alternative approach to the estimation of FPCA using penalized rank 1 approximation to the data matrix is proposed in [20]. Two different versions of smoothed FPCA based on penalized splines with B-splines are discussed in [1].

There are many important applications of smoothed FPCA. The penalized-components functional version of principal component regression and partial least squares are introduced in [32] based on smoothed FPCA. Luo et al. [26] apply smoothed FPCA for testing association of the entire allelic spectrum of genetic variation. Proximity measures between functional mathematical objects are crucial in semi-parametric and nonparametric FDA. In some situations, semi-metric spaces are better adapted than metric spaces for FDA. As motivated in Section 3.4 of Ferraty and Vieu [12], we can use FPCA as a tool to build a class of semi-metrics. Section 4.3 of [2] exploits semi-functional partial linear modeling, involving functional principal component semi-metrics, to forecast electricity consumption data.

The method for smoothed FPCA in Silverman [35] is an important approach in many ways; see Qui and Zhao [29] for detailed discussion. This method can be studied comprehensively using operator theory in Hilbert spaces. Qui and Zhao [29] discuss some theoretical properties of Silverman's method using tools in L^2 and Sobolev spaces. However, we can generalize Silverman's method to an abstract Hilbert space and use perturbation theory to study its theoretical properties in a more general manner.

In this paper, we propose a new approach to study the asymptotic properties of Silverman's smoothed functional principal components in an abstract separable Hilbert space. Our arguments are related to those in Dauxois et al. [9] and involve both Cauchy contours and resolvents. We obtain asymptotic properties using results on the perturbed eigenvalues and eigenvectors of a sample smoothed covariance operator. Consistency and asymptotic distributions of the estimators are derived under mild conditions. For the sake of simplicity of presentation, first we restrict our attention to the first smoothed functional principal component and then extend the same method to the first K principal components.

The paper is set out as follows. In Section 2, we give notations, definitions, assumptions, and the detailed background. Section 3 is devoted to define Silverman's method in an abstract separable Hilbert space and to review some properties. Our main results concerning the asymptotic properties of smoothed functional principal components are given in Section 4. Outlines of the proofs of the lemmas and the theorems in Section 4 are given in Section 5.

2. Theoretical framework

We present notations, definitions, assumptions, and theoretical background that are used throughout. Let $\mathbb{H}$ stand for an infinite-dimensional separable Hilbert space over the real numbers with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. The class of all bounded linear operators mapping $\mathbb{H}$ into itself is a Banach space, denoted by $\mathcal{L}$, with operator norm $\|\cdot\|_{\mathcal{L}}$. The subspace of $\mathcal{L}$ which contains Hilbert–Schmidt operators is denoted by $\mathcal{L}_{HS}$, with norm $\|\cdot\|_{HS}$. A simple example of an operator in $\mathcal{L}_{HS}$ is the operator $(a \otimes b)$, with $a, b \in \mathbb{H}$, defined by its action:

$$\forall_{x \in \mathbb{H}} \quad (a \otimes b)x = \langle x, b \rangle a.$$

A family of orthogonal projections $\{E(t)_{t \in \mathbb{R}}\}$ on Hilbert space $\mathbb{H}$ is called a resolution of identity supported by the compact interval $[m, M]$ if

1. $\mathrm{Im}E(s) \subset \mathrm{Im}E(t)$ whenever $s \le t$,
2. $\mathrm{Im}E(s) = \cap\{\mathrm{Im}E(t) : t > s\}$,
3. $E(t) = 0$ if $t < m$,
4. $E(t) = I$ if $t > M$,

where $\mathrm{Im}E(t)$ represents the image of $E(t)$; see Section V.3 in [14] for details. Let $A \in \mathcal{L}$ be a Hermitian operator. For $t \in \mathbb{R}$, let $E(t)$ be the orthogonal projection of $\mathbb{H}$ onto the spectral subspace of A associated with $(-\infty, t]$. Then, according to Theorem 3.2 in [14], $\{E(t)\}_{t \in \mathbb{R}}$ is a resolution of identity supported by the interval $[m(A), M(A)]$, where

$$m(A) = \inf_{\|f\|=1} \langle Af, f \rangle, \qquad M(A) = \sup_{\|f\|=1} \langle Af, f \rangle.$$

Furthermore, as stated in Theorem 2.1 of [14], $\sigma(A) \subset [m(A), M(A)]$, where $\sigma(A)$ is the spectrum of the operator A.