Journal of Multivariate Analysis 155 (2017) 180–186

Contents lists available at ScienceDirect

Journal of Multivariate Analysis

journal homepage: www.elsevier.com/locate/jmva

Stationary Gaussian Markov processes as limits of stationary autoregressive time series

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ARTICLE INFO

Article history:
Received 26 June 2016
Available online 3 January 2017

AMS 2010 subject classifications:
primary 60G10
secondary 60G15

Keywords:
Continuous autoregressive processes
Stationary Gaussian Markov processes
Stochastic differential equations

ABSTRACT

We consider the class, $\mathbf{C}_p$, of all zero mean stationary Gaussian processes, $\{Y_t : t \in (-\infty, \infty)\}$ with p derivatives, for which the vector valued process $\{(Y_t^{(0)}, \ldots, Y_t^{(p)}) : t \geq 0\}$ is a $p+1$-vector Markov process, where $Y_t^{(0)} = Y(t)$. We provide a rigorous description and treatment of these stationary Gaussian processes as limits of stationary AR(p) time series.

1. Introduction

In many data-driven applications in both the natural sciences and in finance, time series data are often discretized prior to analysis and are then formulated using autoregressive models. The theoretical and applied properties of the convergence of discrete autoregressive ("AR") processes to their continuous analogs (continuous autoregressive or "CAR" processes) have been well studied by many mathematicians, statisticians, and economists; see, e.g., [1,2,4,8,12]. For references on stochastic differential equations, which underlie the theory of CAR processes, we refer to [6,7,14,15].

A special class of autoregressive processes are the discrete-time zero-mean stationary Gaussian Markovian processes on $\mathbb{R}$. The continuous time analogs of these processes are documented in Chapter 10 of [11] and in pages 207–212 of [13]. For processes in this class, the sample paths possess $p-1$ derivatives at each value of t, and the evolution of the process following t depends in a linear way only on the values of these derivatives at t. Notationally, we term such a process as a member of the class $\boldsymbol{C}_p$. For convenience, we will use the notation CAR(p) $= \boldsymbol{C}_p$. The standard Ornstein–Uhlenbeck process is of course a member of $\boldsymbol{C}_1$, and hence CAR(p) processes can be described as a generalization of the Ornstein–Uhlenbeck process.

It is well understood that the Ornstein–Uhlenbeck process is related to the usual Gaussian AR(1) process on a discrete-time index, and that an Ornstein–Uhlenbeck process can be described as a limit of appropriately chosen AR(1) processes; see [7]. In an analogous fashion we show that processes in $\boldsymbol{C}_p$ are related to AR(p) processes and can be described as limits of an appropriately chosen sequence of AR(p) processes.

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http://dx.doi.org/10.1016/j.jmva.2016.12.008

Of course, there is also extensive literature on the weak convergence of discrete-time time series processes, particularly that of ARMA and GARCH processes. For example, Duan [5] considers the diffusion limit of an augmented GARCH process and Lorenz [9] discusses (in Chapter 3) limits of ARMA processes. However, to the best of our knowledge, none of these references (nor simplifications of their results) discuss how to correctly approximate $\boldsymbol{C}_p$ by discrete AR(p) processes, and thus this is the goal of our paper.

Section 2 begins by reviewing the literature on CAR(p) processes, recalling three equivalent definitions of the processes in $\boldsymbol{C}_p$. Section 3 discusses how to correctly approximate $\boldsymbol{C}_p$ by discrete AR(p) processes. The Appendix contains the proof of our main result, Theorem 3.2.

2. Equivalent descriptions of the class $\boldsymbol{C}_p$

There are three distinct descriptions of processes comprising the class $\boldsymbol{C}_p$, which are documented on p. 212 of [13] but in different notation. On pp. 211–212, Rasmussen and Williams [13] prove that these descriptions are equivalent ways of describing the same class of processes. The first description matches the heuristic description given in the introduction. The remaining descriptions provide more explicit descriptions that can be useful in construction and interpretation of these processes. In all the descriptions $Y = \{Y(t) : t \in [0, \infty)\}$ symbolizes a zero-mean Gaussian process on $[0, \infty)$.

In the present paper we use the first of the three equivalent descriptions in [13], as follows. Let Y be stationary. The sample paths are continuous and are $p-1$ times differentiable, a.e., at each $t \in [0, \infty)$. (The derivatives at $t = 0$ are defined only from the right. At all other values of t, the derivatives can be computed from either the left or the right, and both right and left derivatives are equal.) For each $i \in \{1, \ldots, p-1\}$, we denote the ith derivative at t by $Y^{(i)}(t)$. At any $t_0 \in (0, \infty)$, the conditional evolution of the process $\{Y(t) : t \in [0, t_0]\}$ depends in a linear way only on the set of values $\{Y^{(i)}(t_0) : i \in \{0, \ldots, p-1\}\}$. The above can be formalized as follows: let $\{(Y_t^{(0)}, \ldots, Y_t^{(p-1)}) : t \geq 0\}$ denote the values of a mean zero Itô vector diffusion process defined by the system of equations

$$\begin{aligned} dY_t^{(i-1)} &= Y_t^{(i)} dt, \quad t > 0,\ i = 1, \ldots, p-1 \\ dY_t^{(p-1)} &= \sum_{i=0}^{p-1} a_{i+1} Y_t^{(i)} dt + \sigma dW_t \end{aligned} \tag{2.1}$$

for all $t > 0$, where W_t is the Wiener process, $\sigma > 0$. Then let $Y(t) = Y_t^{(0)}$.

2.1. Characterization of stationarity via (2.1)

The system in (2.1) is linear. Stationarity of vector-valued processes described in such a way has been studied elsewhere; see in particular Theorem 5.6.7 on p. 357 in [7]. The coefficients in (2.1) that yield stationarity can be characterized via the characteristic polynomial of the matrix Λ, where $|\Lambda - \lambda I|$ is

$$\lambda^p - a_p \lambda^{p-1} - \cdots - a_2 \lambda - a_1 = 0. \tag{2.2}$$

The process is stationary if and only if all the roots of Eq. (2.2) have strictly negative real parts.

In order to discover whether the coefficients in (2.1) yield a stationary process it is thus necessary and sufficient to check whether all the roots of Eq. (2.2) have strictly negative real parts. In the case of $\boldsymbol{C}_2$ the condition for stationarity is quite simple, namely that a_1, a_2 should lie in the quadrant $a_1 < 0, a_2 < 0$. The covariance functions for $\boldsymbol{C}_2$ can be found in [11, p. 326]. For higher order processes the conditions for stationarity are not so easily described. Indeed, for $\boldsymbol{C}_3$ it is necessary that $a_1 < 0, a_2 < 0, a_3 < 0$ simultaneously, but the set of values for which stationarity holds is not the entire octant. For larger p one needs to study the solutions of the higher order polynomial in Eq. (2.2).

3. Weak convergence of the h-AR(2) process to CAR(2) process

3.1. Discrete time analogs of the CAR processes

We now turn our focus to describing the discrete time analogs of the CAR processes and the expression of the CAR processes as limits of these discrete time processes. In this section, we discuss the situation for $p = 2$. Define the h-AR(2) processes on the discrete time domain $\{0, h, 2h, \ldots\}$ via

$$X_t = b_1^h X_{t-h} + b_2^h X_{t-2h} + \varsigma^h Z_t, \tag{3.1}$$

with $Z_t \sim$ IID $\mathcal{N}(0, 1)$ for all $t = 2h, 3h, \ldots$. The goal is to establish conditions on the coefficients b_1^h, b_2^h and ς^h so that these AR(2) processes converge to the continuous time CAR(2) process as in the system of equations given in (2.1). We then discuss some further features of these processes.