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A natural parametrization of multivariate distributions with limited memory

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ABSTRACT

A unified formulation of the theory of d-variate wide-sense geometric ($\mathcal{G}_d^w$) and Marshall–Olkin exponential ($\mathcal{MO}_d$) distributions is presented in which d-monotone set functions occupy a central role. A semi-analytical derivation of $\mathcal{G}_d^w$ and $\mathcal{MO}_d$ distributions is deduced directly from the lack-of-memory property. In this context, the distributions are parametrized with d-monotone and d-log-monotone set functions arising from the univariate marginal distributions of minima and the d-decreasingness of the survival functions. In addition, a one-to-one correspondence is established between d-monotone (resp. d-log-monotone) set functions and d-variate (resp. d-variate min-infinitely divisible) Bernoulli distributions. The advantage of such a parametrization is that it makes the distributions highly tractable. As a showcase, we derive new results on the minimum stability and divisibility of the $\mathcal{G}_d^w$ family, and on the marginal equivalence in minima of $\mathcal{G}_d^w$ and distributions with geometric minima. Similarly, a surprisingly simple proof is given of the prominent result of Esary and Marshall (1974) on the marginal equivalence in minima of multivariate exponential distributions.

1. Introduction

A random vector $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_d)$ in $\mathbb{N}^d$ is said to follow a wide-sense geometric ($\mathcal{G}_d^w$) distribution if it satisfies the lack-of-memory (LM) property

$$\Pr\big(\tau_i > t_i + s : i \in I\big) = \Pr\big(\tau_i > t_i : i \in I\big)\ \Pr\big(\tau_i > s : i \in I\big) \tag{1}$$

for all non-empty subsets I of $\mathscr{I}_d = \{1, \ldots, d\}$ and all $\mathbf{t} \in \mathbb{N}_0^d$, $s \in \mathbb{N}_0$, where $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Further, a random vector $\boldsymbol{\tau}$ supported on $\mathbb{R}^d_{>0}$ is said to follow a Marshall–Olkin ($\mathcal{MO}_d$) distribution if it satisfies the LM property (1) for all non-empty $I \subseteq \mathscr{I}_d$ and all $\mathbf{t} \in \mathbb{R}_+^d$, $s \in \mathbb{R}_+$, where $\mathbb{R}_{>0} = (0, \infty)$ and $\mathbb{R}_+ = [0, \infty)$.

Although the $\mathcal{G}_d^w$ and $\mathcal{MO}_d$ distributions are generally associated with classical shock models, they can also be viewed as the distributions of minima of subsets of components of a d-variate vector. This alternative characterization first arose as a by-product of complicated computations [9,13] and later on more explicitly from looking at the distribution of first-passage-time models [16,18,25]. In this paper, we revisit these issues and, as a result, we shed new light on the distributions satisfying the LM property.

Let a random vector $\boldsymbol{\tau}$ follow either a $\mathcal{G}_d^w$ or an $\mathcal{MO}_d$ distribution. For each $I \subseteq \mathscr{I}_d$, let

$$\mu(I) = \Pr\Big(\min_{i \in I} \tau_i > 1\Big)$$

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be the univariate distribution of the minimum of the components of $\boldsymbol{\tau}$ belonging to the set I. The collection $\{\mu(I) : I \subseteq \mathscr{S}_d\}$ may then be regarded as the parameters of the $\mathcal{G}_d^{\mathcal{W}}$ or the $\mathcal{MO}_d$ distribution, as the case may be. In particular, note that thanks to the LM property, the survival function of $\boldsymbol{\tau}$ factors into the product of the $\mu(I)$'s. As a result, the d-decreasingness of the survival function reduces to the d-decreasingness of the survival function of a d-variate Bernoulli distribution, or, equivalently, to the monotonicity of the set function μ.

As will be seen herein, the family of $\mathcal{G}_d^{\mathcal{W}}$ distributions is characterized by d-monotone set functions but $\mathcal{MO}_d$ distributions require a stronger concept of monotonicity, namely d-log-monotonicity; see Definition 2.1 and Propositions 2.4 and 3.7. These two monotonicity concepts are related via the d-monotonicity of the set functions $\sqrt[n]{\mu}$ which translates into minimum divisibility properties of the distributions; see Proposition 2.7 and Corollary 3.5.

What emerges from these observations is that resorting to the $\mu(I)$'s as parameters of the $\mathcal{G}_d^{\mathcal{W}}$ and $\mathcal{MO}_d$ survival functions leads to a unified formulation of these distributions, for which a simple probabilistic interpretation can be given; see Theorems 3.1, 3.3, and Remarks 3.2, 3.8. This framework has several advantages; in particular, it will be seen here that:

(a) It can be used to investigate the intricate nature of the $\mathcal{G}_d^{\mathcal{W}}$ distribution and its relationship to the $\mathcal{MO}_d$ distribution. We establish that the $\mathcal{MO}_d$ distributions and the family of min-infinitely divisible $\mathcal{G}_d^{\mathcal{W}}$ distributions are isomorphic and explain why not all $\mathcal{G}_d^{\mathcal{W}}$ distributions have a continuous counterpart; see Theorem 5.4 and the discussion thereafter.
(b) It can be used to prove the main result of [10], viz. the equivalence in minima of $\mathcal{MO}_d$ distributions and distributions with exponential minima, and derive its discrete analogues; see Theorems 7.2 and 7.3.

As an interesting by-product of our study, we obtain a way to construct d-variate min-infinitely divisible Bernoulli distributions from $2^d - 1$ non-negative numbers; see Corollary 4.3.

Results on d-dimensional distributions with the LM property found in the literature are mostly associated with combinatorial difficulties and elaborate theories; see, e.g., [13,16–18,10,23]. In contrast, our work shows that, thanks to the analytical character of the parameters, it is possible to keep both the combinatorial complexity at bay and the level of sophistication elementary.

Before proceeding, we note that throughout the paper we have relegated the proofs to a series of Appendices.

2. Monotone set functions

This section introduces the concept of d-monotonicity for functions defined on the subsets of $\mathscr{S}_d$. As we shall see in Section 3, such functions parametrize discrete and continuous distributions satisfying the LM property.

Let μ be a real-valued function defined on the subsets of $\mathscr{S}_d$. For any $i \in \overline{T}$, where $T \subseteq \mathscr{S}_d$ and $\overline{T}$ denotes the complement of T in $\mathscr{S}_d$, let ∇_i denote the difference operator

$$\nabla_i\, \mu(T) = \mu(T) - \mu\big(T \cup \{i\}\big).$$

For each non-empty subset I of $\overline{T}$, say $I = \{i_1, \ldots, i_{|I|}\}$, define recursively

$$\nabla_I\, \mu(T) = \nabla_{i_{|I|}} \cdots \nabla_{i_1}\, \mu(T),$$

and set $\nabla_\emptyset\, \mu(T) = \mu(T)$. By induction, it is readily seen that

$$\nabla_I\, \mu(T) = \nabla_{i_{|I|}} \cdots \nabla_{i_2} \Big\{ \mu(T) - \mu\big(T \cup \{i_1\}\big) \Big\} = \sum_{J \subseteq I} (-1)^{|J|}\, \mu\big(T \cup J\big). \tag{2}$$

Definition 2.1 (*d-Monotone Set Functions*). A real-valued function μ defined on the subsets of $\mathscr{S}_d$ is said to be d-monotone (non-increasing) if it satisfies

$$\nabla_I\, \mu(T) \geq 0 \tag{3}$$

for all $T \subseteq \mathscr{S}_d$ and all $\emptyset \neq I \subseteq \overline{T}$. By extension, μ is said to be d-log-monotone if it is positive, i.e., $\mu(T) > 0$ for all $T \subseteq \mathscr{S}_d$, and if the composite function $\ln \circ \mu$ is d-monotone.

A function μ, defined on sets that are partially ordered by inclusion, satisfying the condition $\nabla_i\, \mu(T) \geq 0$ for all T and all $i \in \overline{T}$, is referred to in the literature as monotone (non-increasing), or, equivalently, order-reversing. Definition 2.1 is simply a higher-order extension of the monotonicity concept for set functions. In the context of distributions with LM property, d-monotone set functions generalize the notion of d-monotone sequences, which was used in [16,18] to characterize exchangeable subclasses of $\mathcal{G}_d^{\mathcal{W}}$ and $\mathcal{MO}_d$ distributions. Indeed, d-monotone functions on the subsets of $\mathscr{S}_d$ can be viewed as monotone multi-sequences with subscripts $\mathbf{t} \in \{0, 1\}^d$.

Because of some redundancy, the set of conditions in (3) can be reduced to comprise only $2^d - 1$ conditions.

Lemma 2.2 (*Reduced Conditions*). *A real-valued function μ defined on the subsets of $\mathscr{S}_d$ is d-monotone if and only if, for all $T \subsetneq \mathscr{S}_d$, it satisfies the condition $\nabla_{\overline{T}}\, \mu(T) \geq 0$.*