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Multivariate elliptical truncated moments

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ABSTRACT

In this study, we derive analytic expressions for the elliptical truncated moment generating function (MGF), the zeroth-, first-, and second-order moments of quadratic forms of the multivariate normal, Student's t, and generalized hyperbolic distributions. The resulting formulas were tested in a numerical application to calculate an analytic expression of the expected shortfall of quadratic portfolios with the benefit that moment based sensitivity measures can be derived from the analytic expression. The convergence rate of the analytic expression is fast – one iteration – for small closed integration domains, and slower for open integration domains when compared to the Monte Carlo integration method. The analytic formulas provide a theoretical framework for calculations in robust estimation, robust regression, outlier detection, design of experiments, and stochastic extensions of deterministic elliptical curves results.

1. Introduction

The first results on truncated moments were concerned with the linear truncated multivariate normal (MVN) distribution. They were provided by Tallis [47], who later extended his results of linear truncations to the case of elliptical and radial truncation [48]. Tallis [49] built on previous results to compute the moments of a normal distribution with a plane truncation. Mantegna and Stanley [33] used a truncated Lévy distribution to create a distribution where the sums have slow convergence towards the normal, providing first- and second-order moments. Masoom and Nadarajah [34] calculated the truncated moments of a generalized Pareto distribution. Arismendi [3] generalized the results of Tallis [47] for higher-order moments, and for other elliptical distributions such as the Student's t and lognormal distributions, and for a finite mixture of multivariate normal distributions. For a review of truncated moments for different continuous distributions, see [29].

In this study, we derive analytical formulas for the calculation of the elliptical truncated moments of the multivariate normal distribution. We compute an analytical expansion of the elliptical truncated moment generating function (MGF), and

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then derive this expression for the calculation of the elliptical truncated moments. Previous results on elliptical and radial truncated moments on multivariate normal distributions were provided by Tallis [48]. Here, we use the results of Ruben [41] to derive the analytical expressions. We then apply the multivariate normal results to derive the multivariate Student's t (MST) and the multivariate generalized hyperbolic (MGH) elliptical truncated moments. Our results can be considered an extension of those of Ruben [41] for the MST and MGH cases. The importance of elliptical truncated moments' expansions are evident in applications such as the design of experiments [13,50], robust estimation [19], outlier detection [15,38], robust regression [39,51], robust detection [16], risk averse selection [27], and statistical estates' estimation [46].

Other fields where results on elliptical truncated moments can be successfully applied are physics and dynamical systems. Although the results in these areas are generally for deterministic functions, in recent years advances in elliptical curves have attracted the attention of important researchers. Melander et al. [36], Waltz et al. [52], and Ngan et al. [37] provide examples of applications where the extension from deterministic to stochastic elliptical functions can benefit from elliptical truncated moments' results.

This paper makes three contributions: First, we compute an analytic expression for the MGF of the elliptical truncated zeroth- (probability), first-, and second-order moments of the MVN, MST, and MGH distributions. At the time of producing this research, it was the first time that this analytic expression for the moment generating function had been derived. The univariate generalized hyperbolic (UGH) distribution is defined in Barndorff-Nielsen [6] as a variance–mean mixture of a normal distribution and a generalized inverse Gaussian (GIG) distribution, and its properties and applications are studied further in Barndorff-Nielsen and Blaesild [8]. In Barndorff-Nielsen et al. [9], the UGH distribution is extended to the MGH case. The MGH distribution was introduced in finance by Eberlein and Keller [22], Barndorff-Nielsen [7], and Eberlein et al. [23]. An extensive study of the use of the MGH distribution in finance can be found in Eberlein [21].

Second, the results provide use and extend the theory of multivariate truncated moments, as a generalization that could be used to complement other calculations in applied fields. For example, the expected shortfall is the first moment of the distribution truncated at the losses greater than the Value-at-Risk (VaR); the value of a plain-vanilla option is the first moment of the risk-neutral density truncated at the strike price. This generalization of the concept of truncated moments allows us to use the results from one area of finance, such as option theory, to others such as risk management, and vice versa. The first results on the first two-order moments of the MGH distribution are due to Schmidt et al. [44], and were then extended to higher-order moments by Scott et al. [45]. Broda [12] presented some results on truncated moments of the MGH distribution, extending the results of Imhof [28], based on a numerical method of the inversion of the characteristic function. The results of our research complement Broda [12], as the analytic expression we provide is based on the results on moments of the GIG distribution, that are Bessel functions of the first and second kind, for which there exist analytic expressions such as in Mehrem et al. [35].

Third, as a numerical application we provide an analytic expression for the calculation of the elliptical truncated moments of mixtures of multivariate random variables. Expressions for the expected shortfall in the cases of the MVN, MST, and MGH distributions are provided, complementing the results of Broda [11] on heavy-tailed distributions; in turn, his results are an extension of those of Glasserman et al. [26], from using the VaR to using the expected shortfall as a risk measure. In the case of elliptical distributions, Kamdem [30] calculated the VaR and the expected shortfall of a quadratic portfolio for a mixture of elliptical distributions by an integral equation and Yueh and Wong [53] provided analytic expressions for VaR and the expected shortfall when the risk factors are normally distributed by means of a Fourier transform. Our results improve on Kamdem [30] and Yueh and Wong [53], as we provide an analytic expression which is faster to calculate.

The structure of this paper is as follows: Section 2 presents an introduction to quadratic forms in finance and a risk measurement application. Section 3 develops an analytic expression of the expected shortfall in the case of MVN distributions. Section 4 derives the extension of Section 3, for distributions that are mixture with the normal distribution. In Section 5, the analytic expression for the expected shortfall in the case of MGH distributions is presented. In Section 6 a numerical application and the results are presented. Section 7 deals with extreme numerical cases and Section 8 presents our conclusions.

2. Quadratic forms in finance: The expected shortfall as a truncated moment

The heavy-tailedness and the asymmetry of assets' returns motivated the use of non-normal distributions for modeling risks. The Value-at-Risk (VaR) was adopted in the 1990s as the standard risk measure by the industry. However, there were several criticisms of the use of VaR as a risk measure for non-normal (and non-convex) distributions, Artzner et al. [5] then developed a framework for defining coherent risk measures. The expected shortfall, also known as conditional value-at-risk (CVaR) or expected tail loss (ETL), is a risk measure that satisfies the axioms for 'coherent' risk measures. (Rockafellar and Uryasev [40] present a definition of the expected shortfall for general loss distributions in finance, for cases where the distribution is non-continuous.)

Three approaches were used to solve the non-normal modeling of risks: (i) application of copula theory, (ii) non-parametric, and (iii) more general non-elliptical parametric distributions such as Lévy, stable, Pareto, and Pearson distributions. The first approach was motivated by the results of Klar [31], and in Cherubini et al. [18] there is a complete exposition of their application in finance. Nevertheless, except in a limited number of cases as described, e.g., in Genest and Nešlehová [24,25], the copula modeling approach produces multivariate distributions where a closed form of the joint density is unknown. Salem and Mount [42], Madan and Seneta [32], Aït-Sahalia and Lo [2], Scaillet [43], and Chen