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An offspring of multivariate extreme value theory: The max-characteristic function

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ABSTRACT

This paper introduces max-characteristic functions (max-CFs), which are an offspring of multivariate extreme-value theory. A max-CF characterizes the distribution of a random vector in $\mathbb{R}^d$, whose components are nonnegative and have finite expectation. Pointwise convergence of max-CFs is shown to be equivalent to convergence with respect to the Wasserstein metric. The space of max-CFs is not closed in the sense of pointwise convergence. An inversion formula for max-CFs is established.

1. Introduction

Multivariate extreme-value theory (MEVT) is the proper toolbox for analyzing several extremal events simultaneously. Its practical relevance in particular for risk assessment is, consequently, obvious. But on the other hand MEVT is by no means easy to access; its key results are formulated in a measure theoretic setup; a common thread is not visible.

Writing the 'angular measure' in MEVT in terms of a random vector, however, provides the missing common thread: Every result in MEVT, every relevant probability distribution, be it a max-stable one or a generalized Pareto distribution, every relevant copula, every tail dependence coefficient etc. can be formulated using a particular kind of norm on multivariate Euclidean space, called D-norm; see below. For a summary of MEVT and D-norms we refer to Falk et al. [10], Aulbach et al. [1–5], Falk [9]. For a review of copulas in the context of extreme-value theory, see, e.g., [11].

A norm $\|\cdot\|_D$ on $\mathbb{R}^d$ is a *D-norm*, if there exists a random vector (rv) $\boldsymbol{Z} = (Z_1, \dots, Z_d)$ with $Z_i \ge 0$, $\mathrm{E}(Z_i) = 1$, $1 \le i \le d$, such that

$$\|\boldsymbol{x}\|_D = \mathrm{E}\left\{\max_{1\le i\le d}(|x_i|\,Z_i)\right\}, \quad \boldsymbol{x} = (x_1, \dots, x_d) \in \mathbb{R}^d.$$

In this case the rv $\boldsymbol{Z}$ is called *generator* of $\|\cdot\|_D$. Here is a list of D-norms and their generators:

- $\|\boldsymbol{x}\|_\infty = \max_{1\le i\le d}|x_i|$ is generated by $\boldsymbol{Z} = (1, \dots, 1)$,
- $\|\boldsymbol{x}\|_1 = \sum_{i=1}^d |x_i|$ is generated by $\boldsymbol{Z}$ = random permutation of $(d, 0, \dots, 0) \in \mathbb{R}^d$ with equal probability $1/d$,

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- $\|\boldsymbol{x}\|_\lambda = \left(\sum_{i=1}^d |x_i|^\lambda\right)^{1/\lambda}$, $1 < \lambda < \infty$. Let $X_1, \dots, X_d$ be independent and identically Fréchet-distributed random variables, i.e., $\Pr(X_i \le x) = \exp(-x^{-\lambda})$, $x > 0$, $\lambda > 1$. Then $\boldsymbol{Z} = (Z_1, \dots, Z_d)$ with

$$Z_i = \frac{X_i}{\Gamma(1 - 1/\lambda)}, \quad i = 1, \dots, d,$$

generates $\|\cdot\|_\lambda$. By $\Gamma(p) = \int_0^\infty x^{p-1} e^{-x}\, dx$, $p > 0$, we denote the usual Gamma function.

D-norms are a powerful tool when analyzing dependence in MEVT. The first letter of the word "dependence" is, therefore, the reason for the index D.

The generator of a D-norm is not uniquely determined, even its distribution is not. Let, for example, $X \ge 0$ be a random variable with $\mathrm{E}(X) = 1$ and put $\boldsymbol{Z} = (X, \dots, X)$. Then $\boldsymbol{Z}$ generates $\|\cdot\|_\infty$ as well. However, we can, given a generator $\boldsymbol{Z}$ of a D-norm, design a D-norm in a simple fashion so that it characterizes the distribution of $\boldsymbol{Z}$: consider the D-norm on $\mathbb{R}^{d+1}$

$$(t, \boldsymbol{x}) \mapsto \mathrm{E}\left\{\max\left(|t|, |x_1| Z_1, \dots, |x_d| Z_d\right)\right\}.$$

Then it turns out that the knowledge of this D-norm fully identifies the distribution of $\boldsymbol{Z}$; it is actually enough to know this D-norm when $t = 1$, as Lemma 1.1 shows, and this shall be the basis for our definition of a max-characteristic function.

Lemma 1.1. *Let $\boldsymbol{X} = (X_1, \dots, X_d) \ge \boldsymbol{0}$, $\boldsymbol{Y} = (Y_1, \dots, Y_d) \ge \boldsymbol{0}$ be random vectors with $\mathrm{E}(X_i), \mathrm{E}(Y_i) < \infty$ for all $i \in \{1, \dots, d\}$. If we have for each $\boldsymbol{x} > \boldsymbol{0} \in \mathbb{R}^d$*

$$\mathrm{E}\left\{\max(1, x_1 X_1, \dots, x_d X_d)\right\} = \mathrm{E}\left\{\max(1, x_1 Y_1, \dots, x_d Y_d)\right\},$$

then $\boldsymbol{X} =_d \boldsymbol{Y}$, where "$=_d$" denotes equality in distribution.

Proof. Fubini's theorem implies $\mathrm{E}(X) = \int_0^\infty \Pr(X > t)\, dt$ for any random variable $X \ge 0$. consequently, we have for $\boldsymbol{x} > \boldsymbol{0}$ and $c > 0$

$$\begin{aligned}
\mathrm{E}\left\{\max\left(1, \frac{X_1}{cx_1}, \dots, \frac{X_d}{cx_d}\right)\right\} &= \int_0^\infty 1 - \Pr\left\{\max\left(1, \frac{X_1}{cx_1}, \dots, \frac{X_d}{cx_d}\right) \le t\right\} dt \\
&= \int_0^\infty 1 - \Pr(1 \le t, X_i \le tcx_i,\ 1 \le i \le d)\, dt \\
&= 1 + \int_1^\infty 1 - \Pr\left(X_i \le tcx_i,\ 1 \le i \le d\right)\, dt.
\end{aligned}$$

The substitution $t \mapsto t/c$ yields that the right-hand side above equals

$$1 + \frac{1}{c}\int_c^\infty 1 - \Pr(X_i \le tx_i, 1 \le i \le d)\, dt.$$

Repeating the preceding arguments with Y_i in place of X_i, we obtain for all $c > 0$ from the assumption the equality

$$\int_c^\infty 1 - \Pr(X_i \le tx_i, 1 \le i \le d)\, dt = \int_c^\infty 1 - \Pr(Y_i \le tx_i, 1 \le i \le d)\, dt.$$

Taking right derivatives with respect to c we obtain for $c > 0$

$$1 - \Pr(X_i \le cx_i,\ 1 \le i \le d) = 1 - \Pr(Y_i \le cx_i,\ 1 \le i \le d),$$

and, thus, the assertion. □

Let $\boldsymbol{Z} = (Z_1, \dots, Z_d)$ be a random vector, whose components are nonnegative and integrable. Then we call

$$\varphi_{\boldsymbol{Z}}(\boldsymbol{x}) = \mathrm{E}\left\{\max\left(1, x_1 Z_1, \dots, x_d Z_d\right)\right\}, \quad \boldsymbol{x} = (x_1, \dots, x_d) \ge \boldsymbol{0} \in \mathbb{R}^d,$$

the *max-characteristic function* (max-CF) pertaining to $\boldsymbol{Z}$. Lemma 1.1 shows that the distribution of a nonnegative and integrable random vector $\boldsymbol{Z}$ is uniquely determined by its max-CF.

Some obvious properties of $\varphi_{\boldsymbol{Z}}$ are $\varphi_{\boldsymbol{Z}}(\boldsymbol{0}) = 1$, $\varphi_{\boldsymbol{Z}}(\boldsymbol{x}) \ge 1$ for all $\boldsymbol{x}$ and

$$\varphi_{\boldsymbol{Z}}(r\boldsymbol{x}) \begin{cases} \le r\varphi_{\boldsymbol{Z}}(\boldsymbol{x}) & \text{if } r \ge 1, \\ \ge r\varphi_{\boldsymbol{Z}}(\boldsymbol{x}) & \text{if } 0 \le r \le 1. \end{cases}$$

It is straightforward to show that any max-CF is a convex function and, thus, it is continuous and almost everywhere differentiable; besides, its derivative from the right exists everywhere. This fact will be used in Section 2.2, where we will establish an inversion formula for max-CFs.

When $\boldsymbol{Z}$ has bounded components, we have $\varphi_{\boldsymbol{Z}}(\boldsymbol{x}) = 1$ in a neighborhood of the origin. Finally, the max-CF of $\max(\boldsymbol{Z}_1, \boldsymbol{Z}_2)$ (where the max is taken componentwise) evaluated at $\boldsymbol{x}$ is equal to the max-CF of the vector $(\boldsymbol{Z}_1, \boldsymbol{Z}_2)$ evaluated at the point $(\boldsymbol{x}, \boldsymbol{x})$.