Engineering Analysis with Boundary Elements 37 (2013) 293–300

Contents lists available at SciVerse ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

A boundary element method without internal cells for solving viscous flow problems

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ARTICLE INFO

Article history:
Received 9 October 2011
Accepted 26 September 2012
Available online 3 December 2012

Keywords:
Viscous flow
Navier–Stokes equations
Boundary element method
Radial integration method
Domain integrals

ABSTRACT

In this paper, a new boundary element method without internal cells is presented for solving viscous flow problems, based on the radial integration method (RIM) which can transform any domain integrals into boundary integrals. Due to the presence of body forces, pressure term and the non-linearity of the convective terms in Navier–Stokes equations, some domain integrals appear in the derived velocity and pressure boundary-domain integral equations. The body forces induced domain integrals are directly transformed into equivalent boundary integrals using RIM. For other domain integrals including unknown quantities (velocity product and pressure), the transformation to the boundary is accomplished by approximating the unknown quantities with the compactly supported fourth-order spline radial basis functions combined with polynomials in global coordinates. Two numerical examples are given to demonstrate the validity and effectiveness of the proposed method.

1. Introduction

Boundary element method (BEM) is a well-established numerical method due to its distinct feature that only the boundary of the problem needs to be discretized into elements. In the last few decades, significant developments have been made in numerical analysis of viscous fluid flows using boundary element method [1–11]. However, due to the presence of body forces, pressure term and non-linearity of the convective terms in Navier–Stokes equations, the resulting integral equations include domain integrals [1–11]. In order to evaluate these integrals, the computational domain of the problem needs to be discretized into internal cells. Although the cell-integration scheme can give accurate results, the discretization of the domain into cells makes, to a certain extent, the BEM lose its advantage of only boundary discretization.

To circumvent the deficiency of domain integrals appearing in the integral equations, various methods have been developed to transform domain integrals into equivalent boundary integrals. The most widely used approach is the so-called dual reciprocity method (DRM) presented by Nardini and Brebbdia [12] for solid dynamics. In this method, the domain integrals are transformed to the boundary by expressing the body force effect quantities as a series of prescribed basis functions and using the particular solution derived from the differential operator of the problem with these basis functions. In recent years, some attempts have been made to apply the DRM to the Navier–Stokes equations. Power and Partridge [8] successfully transformed the domain integrals of the convective terms into boundary integrals by employing DRM. Further improvement to this approach refers to the works by Sarler and Kuhn [9], Power and Mingo [10], and Florez et al. [11]. However, it is not an easy task to obtain the particular solution for complicated problems. Moreover, the treatment of different types of domain integrals involved in the same integral equation is quite difficult for the DRM. Recently, an effective transformation method, called the radial integration method (RIM), was proposed by Gao [13,14], which not only can transform any type of domain integrals to the boundary, but also can remove various singularities appearing in the domain integrals [14,15]. The distinct feature of RIM is that it can treat different types of domain integrals appearing in the same integral equation in a unified way since it does not resort to any particular solutions.

Apart from the above transformation methods from the domain integral to a boundary integral, the fast techniques (such as fast multipole method [16–18] and wavelet transform method [18]) have also been used to reduce the complexity of domain integral matrices associated with fluid flow problems. Since both the methods can provide a sparse approximation of the fully populated domain matrix, the computer memory and CPU time requirements can been decreased considerably. However, the domain of the problem still needs to be discretized into internal cells for evaluating the domain integrals.

In this paper, a radial integration boundary element method without internal cells for the analysis of viscous flow problems is

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http://dx.doi.org/10.1016/j.enganabound.2012.09.014

presented based on the former works [1–3]. The domain integrals appearing in velocity and pressure boundary-domain integral equations are transformed into equivalent boundary integrals by employing RIM. The domain integrals consisting of the known body forces are analytically and directly transformed into the boundary, while the transformation of domain integrals including unknown quantities is carried out with the use of the compactly supported fourth-order spline radial basis functions augmented by polynomials to approximate the velocity product and pressure as did in DRM. In Section 2, the basic velocity and pressure boundary-domain integral equations proposed in the literatures [1–3] will be reviewed. The transformation of domain integrals to the boundary using RIM is described in detail in Section 3. Two numerical examples are given in Section 4 to demonstrate the validity of the presented method and is followed by a conclusion in Section 5.

2. Review of complete boundary-domain integral equations in viscous flows [1–3]

The governing differential equations in fluid mechanics can be derived from the conservation laws of mass, momentum and energy. In this paper, the flow is assumed to be under isothermal condition, so conservation of energy is not concerned. The continuity and momentum equations in conservation form can be expressed as:

$$\frac{\partial\rho}{\partial t}+\frac{\partial\rho u_i}{\partial x_i}=0 \tag{1}$$

$$\frac{\partial\rho u_i}{\partial t}+\frac{\partial\rho u_i u_j}{\partial x_j}=\frac{\partial\sigma_{ij}}{\partial x_j}+\rho b_i \tag{2}$$

where, ρ is the fluid density, t the time, x_i the ith component of Cartesian coordinates, u_i the ith velocity component, b_i the body force per unit mass (e.g. the gravity force) and σ_{ij} the stress tensor, and summation convection is adopted for the repeated subscripts i and j. For Newtonian fluids, the constitutive relationship between the stresses and velocities based on Stokes' hypothesis can be expressed as:

$$\sigma_{ij}=-p\delta_{ij}+\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)-\frac{2}{3}\mu\frac{\partial u_k}{\partial x_k}\delta_{ij} \tag{3}$$

in which, p is the pressure, μ the dynamics coefficient of viscosity (constant here) and δ_{ij} the Kronecker delta symbol. On the fluid surface with outward normal n_j, the relationship between the stresses and the traction t_i (force per unit area) can be written as:

$$t_i=\sigma_{ij}n_j \tag{4}$$

Applying the weighted residual formulation to Eq. (2) and integrating over the domain Ω bounded with the boundary Γ, the velocity boundary-domain integral equation for two-dimensional (2D) and three-dimensional (3D) problems can be derived by means of Gauss's theorem and the use of integration by parts as follows [1,2]:

$$\begin{aligned} u_i(x)=&\int_\Gamma u^*_{ij}(x,y)t_j(y)d\Gamma(y)-\int_\Gamma t^*_{ij}(x,y)u_j(y)d\Gamma(y)\\ &-\int_\Gamma u^*_{ij}(x,y)n_k(y)\rho(y)u_j(y)u_k(y)d\Gamma(y)\\ &+\int_\Omega u^*_{ij,k}(x,y)\rho(y)u_j(y)u_k(y)d\Omega(y)\\ &+\int_\Omega u^*_{ij,j}(x,y)p(y)d\Omega(y)+\int_\Omega u^*_{ij}(x,y)\rho(y)b_j(y)d\Omega(y)\\ &-\int_\Omega u^*_{ij}(x,y)\frac{\partial\rho u_j}{\partial t}d\Omega(y) \end{aligned} \tag{5}$$

where x denotes the source point and y the field point, $(\,)_{,k}=\partial(\,)/\partial y_k$. Fundamental solutions appearing in integral equation (5) can be expressed as:

$$u^*_{ij}(x,y)=\begin{cases}\frac{1}{16\alpha\pi\mu}\{7\delta_{ij}\ln(\frac{1}{r})+r,_i r,_j\} & \text{for2D}\\ \frac{1}{16\alpha\pi\mu r}\{7\delta_{ij}+r,_i r,_j\} & \text{for3D}\end{cases} \tag{6}$$

$$t^*_{ij}(x,y)=\frac{-1}{8\alpha\pi r^\alpha}\{3(n_i r_{,j}-n_j r_{,i})+(\beta r_{,i}r_{,j}+3\delta_{ij})n_k r_{,k}\} \tag{7}$$

$$u^*_{ij,k}(x,y)=\frac{-1}{16\alpha\pi\mu r^\alpha}(7\delta_{ij}r_{,k}-\delta_{ik}r_{,j}-\delta_{jk}r_{,i}+\beta r_{,i}r_{,j}r_{,k})=\frac{1}{r^\alpha}\Psi_{ijk} \tag{8}$$

$$u^*_{ij,j}(x,y)=\frac{-3r_{,i}}{8\alpha\pi\mu r^\alpha} \tag{9}$$

where $\alpha=\beta-1$ with $\beta=2$ for 2D and $\beta=3$ for 3D problems, r is the distance from the source point x to the field point y, $r_{,i}=\partial r/\partial y_i=(y_i-x_i)/r$, and

$$\Psi_{ijk}=\frac{-1}{16\alpha\pi\mu}(7\delta_{ij}r_{,k}-\delta_{ik}r_{,j}-\delta_{jk}r_{,i}+\beta r_{,i}r_{,j}r_{,k}) \tag{10}$$

Based on velocity integral equation (5) and continuity equation (1), the pressure boundary-domain integral equation can be derived after much manipulation as follows [3]:

$$\begin{aligned} \frac{3}{4\mu}p(x)=&\int_\Gamma u^*_{ij,i}(x,y)t_j(y)d\Gamma(y)-\int_\Gamma t^*_{ij,i}(x,y)u_j(y)d\Gamma(y)\\ &-\int_\Gamma u^*_{ij,i}(x,y)n_k(y)\rho(y)u_j(y)u_k(y)d\Gamma(y)\\ &+\int_\Omega u^*_{ij,ki}(x,y)\rho(y)u_j(y)u_k(y)d\Omega(y)\\ &+\int_\Omega u^*_{ij,i}(x,y)\rho(y)b_j(y)d\Omega(y)-\int_\Omega u^*_{ij,i}(x,y)\frac{\partial\rho u_j}{\partial t}d\Omega(y)\\ &-\frac{3}{4\beta\mu}\rho(x)u_i(x)u_i(x)+\frac{\partial u_i}{\partial x_i} \end{aligned} \tag{11}$$

where

$$t^*_{ij,i}=\frac{-3}{4\alpha\pi r^\beta}(\delta_{jl}-\beta r_{,j}r_{,l})n_l \tag{12}$$

$$u^*_{ij,i}=\frac{-3r_{,j}}{8\alpha\pi\mu r^\alpha} \tag{13}$$

$$u^*_{ij,ki}=\frac{-3}{8\alpha\pi\mu r^\beta}(\delta_{jk}-\beta r_{,j}r_{,k}) \tag{14}$$

From Eq. (11), it can be seen that the boundary integral involving the kernel $t^*_{ij,i}$ is hyper-singular, when the source point x approaches the field point y. Therefore, the pressure integral equation (11) can only be applied to internal points. For boundary points, based on the traction-recovery method [19], the related formulation for evaluation of the pressure can be derived as [2]:

$$p=-t_n-2\mu\varepsilon_\tau-\frac{3\mu}{4\rho}\left(u_i\frac{\partial\rho}{\partial x_i}+\frac{\partial\rho}{\partial t}\right) \tag{15}$$

where t_n is the traction along the normal direction of the boundary surface and ε_τ the tangential strain rate.

The domain integral included in Eq. (11) with the kernel $u^*_{ij,ki}$ is strongly singular when the source point x approaches the field point y. Special integration technique must therefore be adopted in order to make the integral bounded. To do this, applying the singularity separation technique [3,19] to evaluate the integral yields:

$$\begin{aligned} \int_\Omega u^*_{ij,ki}(x,y)\rho(y)u_j(y)u_k(y)d\Omega(y)=&\int_\Omega u^*_{ij,ki}(x,y)\,\{\rho(y)u_j(y)u_k(y)\\ &-\rho(x)u_j(x)u_k(x)\}\,d\Omega(y)+\rho(x)u_j(x)u_k(x)\int_\Omega u^*_{ij,ki}(x,y)d\Omega(y) \end{aligned} \tag{16}$$