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# Admissibility of linear estimators of the common mean parameter in general linear models under a balanced loss function

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ABSTRACT

In order to investigate linearly admissible estimators of the common mean parameter in general linear models, we introduce and motivate the use of a balanced loss function obtained by combining Zellner's idea of balanced loss (Zellner, 1994) with the unified theory of least squares (Rao, 1973). In classes of homogeneous and non-homogeneous linear estimators, sufficient and necessary conditions for linear estimators of the common mean parameter to be admissible are obtained, respectively. A comparison is then made between linearly admissible estimators and a "truly" unified least square estimator.



## 1. Introduction

The problems inherent in combining the results of independent studies for the purpose of estimating unknown parameters have been considered important in a number of areas, such as biomedical studies and metrology. The literature provides many estimators but they typically pertain to situations where the underlying distributions are independent and normal, and a square error loss function is assumed. We refer readers to [13,17,26,28,31,32], as well as the references therein.

Throughout this study, in the absence of the aforementioned assumptions – namely, independence, normality, and the square error loss function – we consider linear models in which, for every $i \in \{1, \ldots, k\}$,

$$Y_i = X\theta + \varepsilon_i, \tag{1}$$

where $Y_i$ is an $n \times 1$ vector of observations, $X$ is an $n \times p$ known design matrix of rank $p \le n$, $\theta$ is a $p \times 1$ linearly estimable vector of a common mean parameter, and $\varepsilon_i$ is an $n \times 1$ error vector. It is assumed that $Y_i$ and $Y_j$ are uncorrelated whenever $i \ne j$, and that $\varepsilon_i$ has mean 0 and covariance matrix $\sigma_i^2 \Sigma$. Furthermore, we assume that $\Sigma \ge 0$ ($\Sigma \ne 0$) is known and $\theta$ and $\sigma_i^2 > 0$ are unknown parameters.

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Motivated by Zellner's idea of balanced loss [33] and the unified theory of least squares [25], the following balanced loss function is proposed for any estimator $d = d(Y_1, \ldots, Y_k)$ of $\theta$:

$$L(d,\theta) = \sum_{i=1}^{k} \delta_i \{w(Y_i - Xd)^\top T^-(Y_i - Xd) + (1-w)(d-\theta)^\top S(d-\theta)\}$$
$$= w \sum_{i=1}^{k} \delta_i (Y_i - Xd)^\top T^-(Y_i - Xd) + (1-w)(d-\theta)^\top S(d-\theta), \quad (2)$$

where $w$ and $\delta_i$ are known weights that range between 0 and 1, with $\delta_1 + \cdots + \delta_k = 1$, $S$ is a known positive definite matrix, and $T = \Sigma + XUX^\top$ with $U \geq 0$, such that $\mathrm{rk}(T) = \mathrm{rk}(X : \Sigma)$, where $\mathrm{rk}(T)$ and $T^-$ denote the rank and reflexive $g$-inverse of $T$, respectively.

The choice of a common $w$ reflects the relative weight assigned to the goodness of fit of each model and the precision of the estimate. The choice of $\delta_i$ reflects the relative weight of the $i$th model in combining the information. The choices of $w$ and $\delta_i$ essentially depend on the experimenter and the objective of the experiment at hand. The experimenter may select these values in line with his or her past experience with similar types of studies, while bearing in mind their association with the experiment, or some prior information about the same.

When the experiment does not provide sufficient information, a simple choice for the weights is $w = 0.5$ and $\delta_1 = \cdots = \delta_k = 1/k$. Under the latter condition, some authors have studied the admissibility of linear functions of the sample mean and the Bayes estimators of $\theta$ for some distributions. For more details, see, e.g., [8,9,12,27]. A balanced loss function reflects both the goodness of fit of the model and the precision of the estimate; hence, compared to some classical loss functions, it is a more comprehensive and reasonable standard by which to measure the estimate. Moreover, from the viewpoint of the sensitivity of loss functions, a balanced loss function is more sensitive than a square error loss function; see [2] for more details. Since the idea of balanced loss was first proposed by Zellner [33], it has received much attention in various contexts. Readers are referred to [1,3–7,10,14,15,18–21,23,24,29,30].

The present study examines the problem of linearly admissible estimators of $\theta$ in a general situation. According to the unified theory of least squares, $U = I_p$ is frequently selected when $\Sigma \geq 0$, where $I_p$ denotes the identity matrix. In this study, without loss of generality, we restrict our attention to this case. Other situations could be handled in a similar way. For simplicity of exposition, the following homogeneous and non-homogeneous classes of linear estimators will be used:

$$\mathcal{LH} = \left\{ \sum_{i=1}^{k} L_i Y_i : L_i \in \mathbb{R}^{p\times n},\ i = 1, \ldots, k \right\}$$

and

$$\mathcal{LI} = \left\{ \sum_{i=1}^{k} L_i Y_i + \alpha : L_i \in \mathbb{R}^{p\times n}, i = 1, \ldots, k, \alpha \in \mathbb{R}^p \right\}.$$

The remainder of this paper is organized as follows. The main results are presented in Section 2. Section 3 offers some preliminary discussion. Section 4 provides proofs for the main results. In Section 5, we compare the risk functions between linearly admissible estimators and a "truly" unified least squares estimator under the balanced loss function. Examples are given in Section 6 and conclusions are drawn in Section 7.

## 2. Main results

In this section, we present our main results; their proofs are given in Section 4. We begin with a definition of "admissibility".

**Definition 2.1.** Given two estimators $d_1 = d_1(Y_1, \ldots, Y_k)$ and $d_2 = d_2(Y_1, \ldots, Y_k)$ of $\theta$, $d_1$ is said to be better than $d_2$ if $R(d_1; \theta, \sigma^2) \leq R(d_2; \theta, \sigma^2)$ holds for all $(\theta, \sigma^2) \in \Theta$ with strict inequality holding at least for one point in $\Theta$, where $R(d_1; \theta, \sigma^2) = \mathrm{E}\{L(d_1, \theta)\}$ is the risk function of $d_1$ and $\Theta = \{(\theta, \sigma^2) : \theta \in \mathbb{R}^p, \sigma^2 = (\sigma_1^2, \ldots, \sigma_k^2)$ with $\sigma_1^2 > 0, \ldots, \sigma_k^2 > 0\}$ is the parameter space. If there exists no estimator that is better than $d_1$ in a class of estimators, then $d_1$ is said to be admissible in this class.

**Theorem 2.1.** *Under model (1) and loss function (2), $L_1Y_1 + \cdots + L_kY_k$ is admissible in $\mathcal{LH}$ if and only if the following hold simultaneously.*

(a) $L_i\Sigma = L_iX(X^\top T^- X)^{-1}X^\top T^-\Sigma$ *for all* $i \in \{1, \ldots, k\}$.
(b) $\widehat{L_i}\Sigma = \lambda_i \widehat{L}\Sigma$ *for all* $i \in \{1, \ldots, k\}$.
(c) $\widehat{L}X\{(X^\top T^- X)^{-1} - I_p\}X^\top \widehat{L}^\top \leq (1-w)\widehat{L}X\{(X^\top T^- X)^{-1} - I_p\}SB^{-1}$,
(d) $\mathrm{rk}[(LX - I_p)\{(X^\top T^- X)^{-1} - I_p\}] = \mathrm{rk}(LX - I_p)$, *where* $\widehat{L_i} = L_i - w\delta_i B^{-1}X^\top T^-$, $L = L_1 + \cdots + L_k$, $\widehat{L} = L - wB^{-1}X^\top T^-$ *with* $B = wX^\top T^- X + (1-w)S$ *and* $\lambda_i$ *belongs to* $[0, 1]$ *with* $\lambda_1 + \cdots + \lambda_k = 1$.