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Central limit theorems and parameter estimation associated with a weighted-fractional Brownian motion[☆]

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A B S T R A C T

Let $B^{a,b}$ be a weighted-fractional Brownian motion with indexes a and b satisfying $|b| < 1 \wedge (1+a)$, $a > -1$ which is a central Gaussian process such that

$$E\left[B_t^{a,b}B_s^{a,b}\right] = \frac{1+b}{2}\int_0^{s\wedge t} u^a((t-u)^b + (s-u)^b)du.$$

In this paper, we consider the asymptotic normality associated with processes

$$\int_0^t \left(\left(B_{s+\varepsilon}^{a,b} - B_s^{a,b}\right)^2 - t^a \varepsilon^{1+b}\right) ds, \quad t \in [0,T], \quad \varepsilon > 0.$$

As an application we study the asymptotic normality of the estimator of parameter $\sigma > 0$ in stochastic process $X_t = \sigma B_t^{a,b} - \beta \int_0^t X_s ds$ by using the *generalized quadratic variation*.

1. Introduction and main results

As an extension of Brownian motion, Bojdecki et al. (2007) introduced and studied a special class of self-similar Gaussian processes which is called the *weighted-fractional Brownian motion* (weighted-fBm in short). This process arises from occupation time fluctuations of branching particle systems with Poisson initial condition and it preserves many similar properties to the fractional Brownian motion. The weighted-fBm with indexes a and b is a mean zero Gaussian process $B^{a,b} = \{B_t^{a,b}, t \geq 0\}$ with $B_0^{a,b} = 0$ and

$$R_{a,b}(t,s) \equiv E\left[B_t^{a,b}B_s^{a,b}\right] = K_{a,b}\int_0^{s\wedge t} u^a((t-u)^b + (s-u)^b)du \tag{1}$$

for all $s, t \geq 0$, where $K_{a,b} > 0$ is a constant and a, b satisfy the condition

$$a > -1, \ -1 < b < 1, \ |b| < 1 + a.$$

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If $E\left[(B_1^{a,b})^2\right] = 1$, i.e. $K_{a,b} = \frac{1}{2\mathbb{B}(1+a,1+b)}$, the weighted-fBm is called standard, where $\mathbb{B}(\cdot,\cdot)$ is the classical Beta function. Clearly, if $a = 0$, the process coincides with a fractional Brownian motion with Hurst parameter $H = \frac{b+1}{2}$ and $B^{a,b}$ is neither a semi-martingale nor a Markov process unless $b = 0$. Although, $B^{a,b}$ has not stationary increments in general, we have (see, Bojdecki et al., 2007 and Yan et al., 2014)

$$c_{a,b}(t \vee s)^a|t-s|^{b+1} \le E\left[\left(B_t^{a,b} - B_s^{a,b}\right)^2\right] \le C_{a,b}(t \vee s)^a|t-s|^{b+1} \tag{2}$$

for $s, t \ge 0$. This process admits the next properties:

- it is $\delta_1 = \frac{1+a+b}{2}$-self similar;
- it is Hölder continuous of order $\delta_2 = \frac{1}{2}(1+b)$;
- it is long-range dependent for $b > 0$ and short-range dependent for $b < 0$;
- the order of the infinitesimal $\sqrt{E[(B_t^{a,b})^2]} \to 0$ is $\delta_3 = \frac{1+a+b}{2}$.

If $a \neq 0$, we find that $\delta_1, \delta_2, \delta_3$ does not coincide. However, the three indexes are coincident for many famous self-similar processes such as fractional Brownian motion, sub-fractional Brownian motion, bi-fractional Brownian motion and Hermite process. Such differences are trouble for its research.

On the other hand, the increments of weighted-fBm $B^{a,b}$ are not asymptotically stationary, i.e.,

- the limit distribution of the processes $\{h^{-\frac{1}{2}(1+b)}\left(B_{t+h}^{a,b} - B_t^{a,b}\right), t \ge 0\}$ depends on $t > 0$, as h tends to zero.

In fact, we have

$$\lim_{h\downarrow 0} \frac{1}{h^{1+b}} E\left(B_{t+h}^{a,b} - B_t^{a,b}\right)^2 = \frac{2K_{a,b}}{1+b} t^a$$

for all $t > 0$. This is also very different with many famous self-similar processes such as some fore-mentioned Gaussian processes. Thus, it seems important to study the weighted-fBm, as a especial class of Gaussian processes having not asymptotically stationary increments. More works for weighted-fBm can be found in Bojdecki et al. (2008a, b), Shen and Yan (2013), Sun et al. (2017), Yan et al. (2014) and the references therein.

Very recently, in Sun et al. (2017) we studied the so-called quadratic covariation defined by

$$\left[f\left(B^{a,b}\right), B^{a,b}\right]_t^{a,b} := \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{1+b}} \int_0^t \left\{f\left(B_{s+\varepsilon}^{a,b}\right) - f(B_s^{a,b})\right\}\left(B_{s+\varepsilon}^{a,b} - B_s^{a,b}\right) ds^{1+b},$$

provided the limit exists uniformly in probability, where f is Borel function on $\mathbb{R}$, and also discussed some related stochastic analysis questions. As an example, we showed that

$$\frac{1}{\varepsilon^{1+b}} \int_0^t (B_{s+\varepsilon}^{a,b} - B_s^{a,b})^2 ds^{1+b} \longrightarrow C_{a,b} t^{1+a+b}$$

almost surely and in L^2, ε tends to zero. In this paper, motivated by this and Gradinaru and Nourdin (2003), and as an extension to Gao et al. (2017), we consider the central limit theorem associated with processes

$$\int_0^t \left(\left(B_{s+\varepsilon}^{a,b} - B_s^{a,b}\right)^2 - s^a\varepsilon^{1+b}\right) ds, \quad t \in [0,T], \quad \varepsilon > 0. \tag{3}$$

For convenience we take $K_{a,b} = \frac{1+b}{2}$ in this paper. In order to study the asymptotic behavior, we need some conditions on parameters a and b and some estimations associated with weighted-fBm based on these conditions. These conditions and estimations point out the complexity of sample paths of weighted-fBm which is also an important motivation to study weighted-fBm. These conditions depend on the two critical values $a = -\frac{1}{2}$ and $b = \frac{1}{2}$, and they are given as follows.

- Condition A: $a > -\frac{1}{2}, \quad -1 < b < \frac{1}{2}, \quad |b| < 1 + a$;
- Condition B: $a > -\frac{1}{2}, \quad b = \frac{1}{2}$;
- Condition C: $a = -\frac{1}{2}, \quad |b| < \frac{1}{2}$;
- Condition D: $-1 < a < -\frac{1}{2}, \quad |b| < 1 + a$;
- Condition E: $a > -\frac{1}{2}, \quad \frac{1}{2} < b < 1 + a$.

Denote

$$\lambda_b = \int_0^1 \left((1+r)^{1+b} + (1-r)^{1+b} - 2\right)^2 \frac{dr}{r^{4+2b}}$$