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Axial symmetric stationary heat conduction analysis of non-homogeneous materials by triple-reciprocity boundary element method

Yoshihiro Ochiai [a,*], Vladimir Sladek [b], Jan Sladek [b]

[a] Department of Mechanical Engineering, Kinki University, 3-4-1 Kowakae, Higashi-Osaka, Osaka 577-8502, Japan
[b] Institute of Construction and Architecture, Slovak Academy of Science, 845 03 Bratislava, Slovakia

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ABSTRACT

The heat conduction problems in homogeneous media can be easily solved by the boundary element method. The spatial variations of heat sources as well as material coefficients gives rise to domain integrals in integral formulations for solution of boundary value problems in functionally gradient materials (FGM), since the fundamental solutions are not available for partial differential equations with variable coefficients, in general. In this paper, we present the development of the triple reciprocity method for solution of axial symmetric stationary heat conduction problems in continuously non-homogeneous media with eliminating the domain integrals. In this method, the spatial variations of domain "sources" are approximated by introducing new potential fields and using higher order fundamental solutions of the Laplace operator.

1. Introduction

In functionally gradient materials (FGMs), the material coefficients (such as the thermal conductivity, etc.) are represented by continuous functions of spatial coordinates. The pure boundary integral formulation is available if the fundamental solution is known. Thus, the boundary element method is applicable to various boundary value problems in homogeneous media, where the governing equations are given by partial differential equations (PDE) with constant coefficients. In continuously non-homogeneous media, however, the governing equations are given by PDE with variable coefficients and the fundamental solution is not available, in general. Having used the simplified fundamental solutions, one obtains the integral formulation with involving also domain integral which is so called boundary-domain formulation and the boundary elements alone are insufficient for approximation of unknown field variables. Besides the boundary elements certain cells are required for the evaluation of domain integrals [1] in such a formulation, hence, the dimensionality reduction merit of the BEM is lost. A great effort has been expended to convert domain integrals into boundary ones, e.g., the dual-reciprocity method has been developed [2,3]. This approach, however, is not suitable for complicated inhomogeneous problems, because the domain must be divided into subdomains in order to achieve reasonable accuracy. Several other excellent methods have been proposed without internal cells [4–7]. Nevertheless, in all BEM formulations including elastoplastic [15] analysis and unsteady problems [13,14], further study is necessary to avoid internal cells.

Ochiai has proposed the triple-reciprocity method for elimination of domain integrals in isotropic steady heat conduction problems [8,9]. In this method, the spatial variations of domain "sources" are approximated by introducing new potential fields and using higher order fundamental solutions of the Laplace operator. The standard BEM degrees of freedom are completely utilized. The domain integrals are converted to boundary integrals and some additional interior point unknowns are introduced. Since the number of boundary elements much smaller than the number of interior cells, this results in saving of that portion of the CPU time which is needed for creation of the discretized algebraic equations in the triple-reciprocity method as compared with the standard BEM. Highly accurate solutions can be obtained solely by using a few of the higher order fundamental solutions. The high accuracy of the triple-reciprocity BEM consists in more accurate numerical treatment of singular domain integrals involved in standard BEM. In this paper, we developed the triple-reciprocity BEM for axially symmetric stationary heat conduction problems in functionally gradient materials. Making use of the axial symmetry and higher order fundamental solutions for 3-d Laplace operator, the original 3-d problem is converted into the 2-d problem with eliminating the domain integrals of heat sources as well as the gradient term due to material inhomogeneity. The triple-reciprocity method has been developed as an improvement of the multiple-reciprocity method [11,12]. In case of laminated materials, a same method can be used

* Corresponding author.
E-mail addresses: ochiai@mech.kindai.ac.jp (Y. Ochiai), vladimir.sladek@savba.sk (V. Sladek).

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to solve heat conduction. The presented method could be modified to handle 2D or 3D problems in Cartesian coordinate systems.

Besides the triple-reciprocity, one can utilize any numerical technique for treatment of considered domain integrals, e.g. wavelet approximation, fast multiple method, etc. Though the latter has been well developed within the BEM formulation, the triple-reciprocity method is rather small modification of the standard BEM formulation and can be easily implemented into existing BEM codes.

1.1. Main notations

- T — Temperature.
- $\lambda(q)$, $\lambda_1^S(q)$ — Heat conduction coefficient (thermal conductivity); approximation for $\lambda(q)$.
- $\omega\Gamma;(q), W_1{}^N(q)$ — Volume density of heat sources; approximation for $w(q)$.
- Ω, Γ — Domain and its boundary in the plane of axial symmetry.
- r, z — Radial and axial coordinates in cylindrical coordinate system.
- n_r, n_z — Radial and axial components of outer unit normal vector on Γ.
- $\delta\left(q-q_j\right)$ — Dirac delta function.
- $G^{[f]}(p,q)$, $T^{[f]}(p,q)$ — Higher order fundamental solution of the Laplace operator and their integrals with respect to the angular coordinate.
- $K(m)$, $E(m)$ — Complete elliptic integrals of the first and second kind.

2. Axial symmetric steady heat conduction

In this paper, we are concerned with the steady-state axial symmetric heat conduction problem in inhomogeneous materials. The governing equation can be expressed as

$$\tilde{\nabla}\cdot\left\{\lambda(\tilde{q})(\tilde{\nabla}T(\tilde{q}))\right\} = -w(\tilde{q}), \tag{1}$$

where $T(\tilde{q})$ is the temperature at point $\tilde{q}\in V$, $\lambda(\tilde{q})$ is the heat conductivity, and $\tilde{\nabla}$ is the gradient operator in a 3-d coordinate system. The tilde symbol is used for quantities (points or vectors) related to a 3-d coordinate system in order to distinguish them from the quantities related to the plane (r,z) introduced later. The volume density of heat sources $w(\tilde{q})$, is assumed to be a continuous function.

2.1. Conversion of the integral formulation form 3-d into 2-d

In the case axially symmetric problems, the analyzed domain V is generated by rotation of a 2-d domain Ω around the axis z, with Ω lying in the plane involving the revolving axis (Fig. 1).

In axially symmetric problems, it is appropriate to utilize cylindrical coordinate system, where $\tilde{\nabla} = \mathbf{e}_r\partial/\partial r + \mathbf{e}_z\partial/\partial z$, $\tilde{\nabla}^2 = \partial^2/\partial r^2 + (1/r)\partial/\partial r + \partial^2/\partial z^2$. Furthermore, we shall use capital letters for boundary points and prime coordinates for source points in contrast to small letters for interior points and coordinates without prime for field points. Because of the rotational symmetry it is sufficient to know the solution of an axially symmetric boundary value problems in the domain Ω, since all physical fields are independent on the angular variable φ. Therefore the point $\tilde{q}$ in Eq. (1) can be replaced by q, too.

As regards the continuity of the material coefficient $\lambda(\tilde{q}) = \lambda(q)$, we shall distinguish three different classes of problems: (i) $\lambda(q)$ is a continuous and differentiable function within the analyzed domain Ω, (ii) $\lambda(q)$ is a continuous function but $\tilde{\nabla}\lambda$ is discontinuous at certain points q^d in Ω, (iii) $\lambda(q)$ is discontinuous at certain points q^d in Ω.

As long as $\lambda(q)$ is a continuous function, Eq. (1) can be rewritten as

$$\tilde{\nabla}^2 T(q) = -W_1(q) - W_1^N(q), \tag{2}$$

where

$$W_1(q) = \frac{\tilde{\nabla}\lambda(q)\bullet\tilde{\nabla}T(q)}{\lambda(q)}, \quad W_1^N(q) = \frac{w(q)}{\lambda(q)} \tag{3}$$

are uniquely defined at $q\in\Omega$ provided that the gradient of the heat conduction coefficient is continuous. Otherwise, if $W_1(q)$ is discontinuous, a special treatment will be required. If $\lambda(q)$ suffers a discontinuity $[\lambda(q^d)]^*$ at certain points q^d in Ω, the gradients $\tilde{\nabla}\lambda(q)$ give rise to additional point heat sources and a special treatment of Eq. (1) is required because the "sources" specified by Eq. (3) would be non-unique and inapplicable. The problems involving the discontinuities of the heat conduction coefficient and/or its gradients will be discussed separately. Now, we shall deal with problems governed by Eq. (2) with continuous sources given by Eq. (3).

It is well known [1] that the fundamental solution of the Laplace operator in 3-d problems is given as $G^{[1]}\left(p,\tilde{Q}\right) = \left(1/4\pi|p-\tilde{Q}|\right)$, and the integral representation of the temperature field governed by Eq. (2) is

$$c(p)T(p) = \int_S \left[\frac{\partial T}{\partial \mathbf{n}}(Q)G^{[1]}\left(p,\tilde{Q}\right) - T(Q)\frac{\partial G^{[1]}\left(p,\tilde{Q}\right)}{\partial \mathbf{n}\left(\tilde{Q}\right)}\right]dS\left(\tilde{Q}\right) + \int_V \left[W_1(q) + W_1^N(q)\right]G^{[1]}(p,\tilde{q})\,dV(\tilde{q}), \tag{4}$$

where the free-term coefficient $c(p)=1$ as long as p is an interior point, while it depends on the local geometry of the boundary, if $p=P$ lies on the boundary [1,16,22]. Since $dS\left(\tilde{Q}\right) = R'd\varphi' d\Gamma(Q)$, $dV(\tilde{q}) = r'd\varphi' d\Omega(q)$ and only the integral kernels are dependent on the angular variable, this integration can be performed in closed form and (4) results in [16]

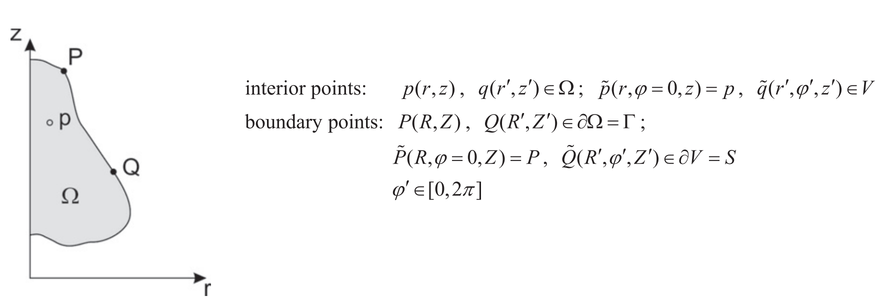

Fig. 1. Sketch of domain Ω with interior and boundary points.