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Non-atomic bivariate copulas and implicitly dependent random variables

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ABSTRACT

Two (continuous) random variables X and Y are *implicitly dependent* if there exist Borel functions α and β such that $\alpha \circ X = \beta \circ Y$ almost surely. The copulas of such random variables are exactly the copulas that are factorizable as the $*$-product of a left invertible copula and a right invertible copula. Consequently, every implicit dependence copula assigns full mass to the graph of $f(x) = g(y)$ for some measure-preserving functions f and g but the converse is not true in general.

We obtain characterizations of a copula C assigning full mass to the graph of $f(x) = g(y)$ in terms of a partial factorizability of its Markov operator T_C and in terms of the non-atomicity of two newly defined associated σ-algebras σ_C and σ_C^*, in which case C is called *non-atomic*. As an application, we give a broad sufficient condition under which a copula with fractal support has an implicit dependence support. Under certain extra conditions, we explicitly compute the left invertible and right invertible factors of the copula with fractal support.

1. Introduction

It is well-known that the bivariate copula of two continuous random variables completely captures their dependence structure. Notable examples are the *independence copula* $\Pi(u, v) = uv$, which corresponds to independent random variables, and the copulas of completely dependent random variables, called *complete dependence copulas*. Since it was discovered (Mikusiński et al., 1992, 1991; Kimeldorf and Sampson, 1978) that there are complete dependence copulas arbitrarily closed to Π in the uniform norm, many norms have been introduced and investigated in the literature (Darsow and Olsen, 1995) giving rise to measures of dependence such as ω in Siburg and Stoimenov (2009) and ζ_1 in Trutschnig (2011). These dependence measures defined in terms of the copula's first partial derivatives attain the maximum value 1 at least for complete dependence copulas and the minimum value 0 when and only when the copula is Π. However, with respect to these dependence measures, the independence copula can still be approximated by *implicit dependence copulas* (Chaidee et al., 2016), defined as copulas of random variables X and Y which are implicitly dependent in the sense that $\alpha \circ X = \beta \circ Y$ a.s. for some Borel measurable functions α and β. For some Rényi-type measures of dependence (Rényi, 1959) such as ω_* in Ruankong et al. (2013) and ν_* in Kamnitui et al. (2015), with respect to which all complete dependence copulas have measure 1, all implicit dependence copulas also attain the maximum measure (Ruankong et al., 2013, Corollary 4.12). It is then evident that implicit dependence copulas play a crucial role in understanding as well as comparing and contrasting measures of MCD and Rényi-type dependence measures. Every implicit dependence copulas assigns its full mass to the graph

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of $f(x) = g(y)$, called an implicit graph, for some measure-preserving functions f and g. Closely related and constituting a much larger class than the implicit dependence copulas are the copulas whose mass is concentrated on an implicit graph.

Motivated by the concept of invariant sets in Darsow and Olsen's study of idempotent copulas (Darsow and Olsen, 2010), we shall investigate copulas C assigning full mass to implicit graphs via their corresponding Markov operators T_C and associated σ-algebras σ_C and σ_C^*. For the copula C of $\mathscr{U}[0,1]$-random variables X and Y, S belongs to σ_C and R belongs to σ_C^* if and only if $Y \in S$ with probability one given that $X \in R$. We derive some fundamental properties of these associated σ-algebras. Intuitively, the larger the σ_C is, the stronger the dependence of Y on X is. We then obtain characterizations of a copula C assigning its full mass to an implicit graph in terms of the factorizability of the corresponding Markov operator T_C on a subclass of the Borel functions and in terms of the size (non-atomicity) of the associated σ-algebras σ_C and σ_C^*. Naturally, these characterizations could be useful in investigating singular copulas. Our main results find an application in copulas with fractal supports introduced by Fredricks, Nelsen and Rodríguez-Lallena (Fredricks et al., 2005). Given a transformation matrix A, there is a unique copula C_A such that $[A](C_A) = C_A$, where $[A]$ maps the class of bivariate copulas into itself according to the weights given by the entries in A. As a consequence, we obtain a broad sufficient condition on a transformation matrix A under which the copula C_A is non-atomic and hence assigns its full mass to an implicit graph. Working directly with the transformation matrix A, a sufficient condition under which C_A is an implicit dependence copula is also given. Our ongoing research is to find a characterization of general implicit dependence copulas via behaviors of their σ-algebras. Such a characterization would be beneficial in the study of products of implicit dependence copulas.

The manuscript is organized as follows. Section 2 lays the necessary background on copulas and Markov operators for the rest of the paper. We then define the associated σ-algebras of a copula and prove their basic properties in Section 3. Section 4 gives a definition of non-atomic copulas and some of their fundamental properties summarizing in characterizations of non-atomic copulas. In the final section, the characterizations are used in an investigation of copulas with fractal support. We also give a sufficient condition on a transformation matrix under which the induced invariant copula can be written as the product of a left invertible copula and a right invertible copula.

2. Background on copulas and Markov operators

Let λ denote the Lebesgue measure on $\mathbb{R}$, $\mathbf{I} \equiv [0,1]$ and $\mathscr{B} \equiv \mathscr{B}(\mathbf{I})$ the Borel σ-algebra on $\mathbf{I}$. Since we always consider λ-integrable functions on $\mathbf{I}$ that are measurable with respect to various sub-σ-algebras $\mathscr{M}$ of $\mathscr{B}$, we will denote by $L^1(\mathscr{M})$ the class of λ-integrable $\mathscr{M}$-measurable functions on $\mathbf{I}$. $\mathscr{B}$-measurable functions are called *Borel functions*. For $A \in \mathscr{B}$, $\mathbf{1}_A$ denotes the indicator function of A and $\mathbf{1} \equiv \mathbf{1}_\mathbf{I}$.

A (bivariate) *copula* C is a function from $\mathbf{I}^2$ to $\mathbf{I}$ which is the joint distribution function of two random variables uniformly distributed on $[0,1]$. For random variables X and Y with joint distribution $F_{X,Y}$ and continuous marginal distributions F_X and F_Y, there exists, by the Sklar's theorem, a unique copula C, called the *copula of X and Y*, for which $F_{X,Y}(x,y) = C(F_X(x), G_Y(y))$ for all x, y. The *independence copula* is the product copula $\Pi(u,v) = uv$. *Complete dependence copulas* are either the copulas $C_{ef} \equiv C_{e,f}$ or $C_{fe} \equiv C_{f,e}$ where $e(x) = x$ and f is a measure-preserving function on $\mathbf{I}$ in the sense that $\lambda(f^{-1}(B)) = \lambda(B)$ for every $B \in \mathscr{B}$. Here, $C_{f,g}(u,v) = \lambda(f^{-1}([0,u]) \cap g^{-1}([0,v]))$ for $u, v \in \mathbf{I}$. The *comonotonic* and *countermonotonic* copulas are $M = C_{e,e}$ and $W = C_{e,\mathbf{1}-e}$, respectively.

Definition 1. Two random variables X and Y are said to be *implicitly dependent* if there exist Borel functions α and β such that $\alpha \circ X = \beta \circ Y$ almost surely. The copula of two implicitly dependent continuous random variables is called an *implicit dependence copula*.

It is evident that all implicit dependence copulas are of the form $C_{ef} * C_{ge}$ for some measure-preserving functions f and g on $\mathbf{I}$.

Each copula C induces a *doubly stochastic measure* μ_C by $\mu_C((a,b] \times (c,d]) = C(b,d) - C(b,c) - C(a,d) + C(a,c)$. The *support* of C is then defined as the support of the induced measure μ_C, i.e. the complement of the union of all open sets having zero μ_C-measure. One can construct a new copula by taking any convex combinations of two or more copulas. Any two copulas C, D also give rise to a new copula via the $*$-*product*: $(C * D)(u,v) = \int_0^1 \partial_2 C(u,t)\partial_1 D(t,v)\,dt$. The binary operation $*$ makes the class of copulas a monoid with null element Π and identity M. If $C * D = M$ then C is a left inverse of D and D is a right inverse of C. The left invertible copulas are exactly the complete dependence copulas C_{ef}, while the right invertible copulas are exactly the complete dependence copulas C_{fe}. See Nelsen (2006) and Durante and Sempi (2015) for comprehensive introductions to many aspects of copulas.

A linear operator T on $L^1(\mathscr{B})$ is called a *Markov operator* if

i. $T\mathbf{1} = \mathbf{1}$,
ii. $\int_0^1 T\psi\,d\lambda = \int_0^1 \psi\,d\lambda$ for every $\psi \in L^1$, which is equivalent to $T^*\mathbf{1} = \mathbf{1}$, and
iii. $T\psi \geq 0$ for every $\psi \geq 0$, which means T is positive.

So a Markov operator must be a bounded linear operator on L^1 (and L^∞). From Olsen et al. (1996), for each copula C, there corresponds a Markov operator T_C defined by

$$(T_C\psi)(x) = \frac{d}{dx}\int_0^1 \partial_2 C(x,t)\psi(t)\,dt.$$