Journal of Statistical Planning and Inference 182 (2017) 29–49

Contents lists available at ScienceDirect

Journal of Statistical Planning and Inference

journal homepage: www.elsevier.com/locate/jspi

Sequential monitoring of the tail behavior of dependent data

Yannick Hoga [a,*], Dominik Wied [b,c]

[a] Faculty of Economics and Business Administration, University of Duisburg–Essen, Universitätsstraße 12, D-45117 Essen, Germany
[b] University of Cologne, Germany
[c] TU Dortmund, Germany

A R T I C L E   I N F O

Article history:
Received 10 May 2016
Received in revised form 16 August 2016
Accepted 16 August 2016
Available online 1 November 2016

JEL classification:
C12
C14
C22

Keywords:
Sequential monitoring
Change point
β-mixing
Tail index
Extreme quantiles
Functional central limit theorem

A B S T R A C T

We construct a sequential monitoring procedure for changes in the tail index and extreme quantiles of β-mixing random variables, which can be based on a large class of tail index estimators. The assumptions on the data are general enough to be satisfied in a wide range of applications. In a simulation study empirical sizes and power of the proposed tests are studied for linear and non-linear time series. Finally, we use our results to monitor Bank of America stock log-losses from 2007 to 2012 and detect changes in extreme quantiles without an accompanying detection of a tail index break.

1. Motivation

The tail index of a random variable is arguably one of the most important parameters of its distribution: It determines some fundamental properties like the existence of moments, tail asymptotics of the distribution and the asymptotic behavior of sums and maxima. As a measure of tail thickness, the tail index is used in fields where heavy tails are frequently encountered, such as (re)insurance, finance, and teletraffic engineering (cf. Resnick, 2007, Sec. 1.3, and the references cited therein). Particularly in finance, the closely related extreme quantiles play a prominent role as a risk measure called Value-at-Risk (VaR).

The use of the variance as a risk measure has a long tradition in finance. Under Gaussianity the variance completely determines the tails of the distribution, which is no longer the case with heavy-tailed data. Hence, in order to assess the tail behavior of a time series, practitioners often estimate the tail index or an extreme quantile, the implicit assumption being their constancy over time. There are several suggestions in the literature on how to test this crucial assumption: Quintos et al. (2001) developed so called recursive, rolling and sequential tests for independent and GARCH data for tail index constancy based on the Hill (1975) estimator. Kim and Lee (2011) investigated their tests for more general β-mixing time series. Taking a likelihood approach for independent data, Dierckx and Teugels (2010) focus on breaks in the tail index for environmental data. Tests based on other estimators than the Hill (1975) estimator were first proposed by Einmahl et al. (2016)

* Corresponding author.
E-mail addresses: yannick.hoga@vwl.uni-due.de (Y. Hoga), dwied@uni-koeln.de (D. Wied).

http://dx.doi.org/10.1016/j.jspi.2016.08.010

for independent and Hoga (2017+b) for dependent data. To the best of our knowledge the only paper dealing with changes in extreme quantiles is Hoga (2017+a). All these tests are of a retrospective nature.

We are not aware of any work on online surveillance methods for constancy of the tail index and extreme quantiles. This is important because, as noted in Chu et al. (1996), '[b]reaks can occur at any point, and given the costs of failing to detect them, it is desirable to detect them as rapidly as possible. One-shot tests cannot be applied in the usual way each time new data arrive, because repeated application of such tests yields a procedure that rejects a true null hypothesis of no change with probability one as the number of applications grows.' This paper will fill this gap for closed-end procedures. To allow for sufficient flexibility in the use of tail index estimators, we will use the approach of Hoga (2017+b).

Whether a monitoring procedure for a change in the tail index or an extreme quantile is of interest will largely be a matter of context. If interest centers on VaR, which is widely used in the banking industry and by financial regulators as a risk measure, the quantile monitoring procedure will be more relevant. If however interest centers on the mean excess function of the (log-transformed) data X, then, since $\mathrm{E}\left(\log X - \log t | X > t\right)$ converges to the extreme value index of X as $t \to \infty$, the tail index alternative seems to be more appropriate. Furthermore, the tail index per se could also be of interest as there are indications that it has predictive power for stock returns (Kelly and Jiang, 2014), where higher (lower) tail indices of returns indicate higher (lower) absolute returns.

The outline of this paper is as follows. The main results under the null and two alternatives are stated in Section 2, where an example of a time series satisfying our assumptions is also given. Simulations and an empirical application are presented in Sections 3 and 4 respectively. All proofs are collected in an Appendix.

2. Main results

2.1. Preliminaries and assumptions

To introduce the required notation let $X_1, \ldots, X_n$ be a sequence of random variables defined on some probability space $(\Omega, \mathcal{A}, P)$ with survivor function $\bar{F}_i(x) := 1 - F_i(x) = P(X_i > x)$, that is regularly varying with parameter $-\alpha_i$ (written $\bar{F}_i \in RV_{-\alpha_i}$), i.e.,

$$\bar{F}_i(x) = x^{-\alpha_i} L_i(x), \quad x > 0, \tag{1}$$

where $L_i : (0, \infty) \to (0, \infty)$ is slowly varying, i.e.,

$$\lim_{x \to \infty} \frac{L_i(\lambda x)}{L_i(x)} = 1 \quad \forall \lambda > 0. \tag{2}$$

If X_i is Pareto distributed, then $L_i(x) \equiv c > 0$. Since slow variation of the function $L_i(x)$ means, loosely speaking, that it behaves like a constant function at infinity, we say that X_i with tails as in (1) has *Pareto-type tails*. In the context of extreme value theory, α_i is called the tail index and $\gamma_i := 1/\alpha_i$ the extreme value index (Resnick, 2007, Sec. 4.5.1).

Define

$$U_i(x) := F_i^{-1}\left(1 - \frac{1}{x}\right), \quad x > 1,$$

as the $(1 - 1/x)$-quantile, F_i^{-1} being the left-continuous inverse of F_i. Then, recall that (1) is equivalent to

$$\frac{U_i(\lambda x)}{U_i(x)} \underset{(x \to \infty)}{\longrightarrow} \lambda^{\gamma} \tag{3}$$

(e.g., Resnick, 2007, Prop. 2.6 (v)). Throughout, $k = k_n \in \mathbb{N}$ will denote a sequence satisfying $k \leq n - 1$,

$$k \underset{(n \to \infty)}{\longrightarrow} \infty \quad \text{and} \quad \frac{k}{n} \underset{(n \to \infty)}{\longrightarrow} 0, \tag{4}$$

controlling the number of upper order statistics used in the estimation of the tail index and $p = p_n \to 0, n \to \infty$, will denote a sequence of small probabilities, for which we want to test for a change in an appertaining extreme (right-tail) quantile $U_i(1/p)$. As is customary in extreme value theory, we will usually drop the subindex n and simply write k and p. For $t - s \geq 1/n$ and $y \in [0, 1]$ set

$$X_k(s, t, y) := (\lfloor k(t-s)y \rfloor + 1)\text{ th largest value of } X_{\lfloor ns \rfloor + 1}, \ldots, X_{\lfloor nt \rfloor}.$$

Under the assumption of strictly stationary X_i we write $\bar{F} = \bar{F}_i$ and $U = U_i$. Let

$$\widehat{\gamma}(s, t) := \widehat{\gamma}_n(s, t), \quad 0 \leq s < t < \infty, t - s \geq 1/n,$$