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Bi-log-concave distribution functions[☆]

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ABSTRACT

Nonparametric statistics for distribution functions F or densities $f = F'$ under qualitative shape constraints constitutes an interesting alternative to classical parametric or entirely nonparametric approaches. We contribute to this area by considering a new shape constraint: F is said to be bi-log-concave, if both $\log F$ and $\log(1 - F)$ are concave. Many commonly considered distributions are compatible with this constraint. For instance, any c.d.f. F with log-concave density $f = F'$ is bi-log-concave. But in contrast to log-concavity of f, bi-log-concavity of F allows for multimodal densities. We provide various characterisations. It is shown that combining any nonparametric confidence band for F with the new shape constraint leads to substantial improvements, particularly in the tails. To pinpoint this, we show that these confidence bands imply non-trivial confidence bounds for arbitrary moments and the moment generating function of F.

1. Introduction

In nonparametric statistics one is often interested in estimators or confidence regions for curves such as densities or regression functions. Estimation of such curves is typically an ill-posed problem and requires additional assumptions. Interesting alternatives to smoothness assumptions are qualitative constraints such as monotonicity or concavity.

Estimation of a distribution function F based on independent, identically distributed random variables $X_1, X_2, \ldots, X_n$ with c.d.f. F is common practice and does not require restrictive assumptions. But nontrivial confidence regions for certain functionals of F such as the mean do not exist without substantial additional constraints (cf. Bahadur and Savage, 1956).

A growing literature on density estimation under shape constraints considers the family of log-concave densities. These are probability densities f on $\mathbb{R}^d$ such that $\log f : \mathbb{R}^d \to [-\infty, \infty)$ is a concave function. For more details see Bagnoli and Bergstrom (2005), Cule et al. (2010), Dümbgen and Rufibach (2009, 2011), Walther (2009), Seregin and Wellner (2010), Dümbgen et al. (2011) and the references cited therein. Most efforts in these papers are devoted to point estimation. Schuhmacher et al. (2011) obtain a nonparametric confidence region by combining the log-concavity constraint and a standard Kolmogorov–Smirnov confidence region. But its explicit computation is difficult, and this is one motivation to search for alternative shape constraints in terms of the distribution function F directly.

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While many popular densities are log-concave, this constraint can be too restrictive in applications with a multimodal density. In the present paper we consider a model with a new and weaker constraint on the distribution function:

Definition (*Bi-log-concavity*)**.** A distribution function F on the real line is called *bi-log-concave* if both $\log F$ and $\log(1-F)$ are concave functions from $\mathbb{R}$ to $[-\infty, 0]$.

Many distribution functions satisfy this constraint. In particular, when F has a log-concave density $f = F'$, it is bi-log-concave Bagnoli and Bergstrom (2005). But indeed, bi-log-concavity of F is a much weaker constraint. As shown later, F may have a density with an arbitrary number of modes. Thus, we consider estimation of distributions under shape constraints for a wider family of distributions.

The remainder of this paper is organised as follows: In Section 2 we present characterisations of bi-log-concavity and explicit bounds for F and its density $f = F'$. In Section 3 we describe exact (conservative) confidence bands for F. They are constructed by combining the bi-log-concavity constraint with standard confidence bands for F such as, for instance, the Kolmogorov–Smirnov band or Owen's (1995) band. A numerical example with the distribution of CEO salaries (Woolridge, 2000) illustrates the usefulness of the proposed method. The benefits of adding the shape constraint are pinpointed in Section 4. It is shown that combining a reasonable confidence band with the new shape constraint leads to non-trivial honest confidence bounds for various quantities related to F. These include its density, hazard function and reverse hazard function, its moment generating function and arbitrary moments. All proofs are deferred to Section 5.

2. Bi-log-concave distribution functions

In what follows we call a distribution function F *non-degenerate* if the set

$$J(F) := \{x \in \mathbb{R} : 0 < F(x) < 1\}$$

is nonvoid. Notice that in the case of $J(F) = \emptyset$ the distribution function F would correspond to the Dirac measure δ_m at some point $m \in \mathbb{R}$, i.e. $F(x) = 1_{[x \geq m]}$.

Our first theorem provides three alternative characterisations of bi-log-concavity which are expressed by different constraints for F and its derivatives.

Theorem 1. *For a non-degenerate distribution function F the following four statements are equivalent:*

(i) *F is bi-log-concave;*

(ii) *F is continuous on $\mathbb{R}$ and differentiable on $J(F)$ with derivative $f = F'$ such that*

$$F(x+t) \begin{cases} \leq F(x) \exp\Big(\dfrac{f(x)}{F(x)}\, t\Big) \\ \geq 1 - (1 - F(x)) \exp\Big(-\dfrac{f(x)}{1-F(x)}\, t\Big) \end{cases} \tag{1}$$

for arbitrary $x \in J(F)$ and $t \in \mathbb{R}$.

(iii) *F is continuous on $\mathbb{R}$ and differentiable on $J(F)$ with derivative $f = F'$ such that the hazard function $f/(1-F)$ is non-decreasing and the reverse hazard function f/F is non-increasing on $J(F)$.*

(iv) *F is continuous on $\mathbb{R}$ and differentiable on $J(F)$ with bounded and strictly positive derivative $f = F'$. Furthermore, f is locally Lipschitz-continuous on $J(F)$ with L^1-derivative $f' = F''$ satisfying*

$$\frac{-f^2}{1-F} \leq f' \leq \frac{f^2}{F}. \tag{2}$$

The set of all distribution functions F with the properties stated in Theorem 1 is denoted as $\mathcal{F}_{\mathrm{blc}}$. The inequalities (2) in statement (iv) can be reformulated as follows: $\log f$ is locally Lipschitz-continuous on $J(F)$ with L^1-derivative $(\log f)'$ satisfying

$$(\log(1-F))' \leq (\log f)' \leq (\log F)',$$

where we remark that the L^1-derivative of a function h on an open interval $J \subset \mathbb{R}$ is a locally integrable function h' on J such that $h(y) - h(x) = \int_x^y h'(t)\,dt$ for all $x, y \in J$.

Example (*Bi-modal Density*)**.** Consider the mixture $2^{-1}\mathcal{N}(-\delta, 1) + 2^{-1}\mathcal{N}(\delta, 1)$ with $\delta > 0$. It can be shown numerically that the corresponding c.d.f. F is bi-log-concave for $\delta \leq 1.34$ but not for $\delta \geq 1.35$. This distribution has a bi-modal density for $\delta = 1.34$. The corresponding c.d.f. F is shown in Fig. 1(a), together with the functions $1 + \log F \leq F \leq -\log(1-F)$, the inequalities following from $\log(1+y) \leq y$ for arbitrary $y \geq -1$. Bi-log-concavity means that the lower bound $1 + \log F$ is concave while the upper bound $-\log(1-F)$ is convex. Figs. 1–2 illustrate the various characterisations of the bi-log-concavity constraint as given in Theorem 1. In particular, Fig. 1(b) shows the bounds from part (ii) for one particular point $x \in J(F)$. Fig. 2(a) shows the density f together with the hazard function $f/(1-F)$ and the reverse hazard function f/F. It is apparent that the latter two satisfy the monotonicity properties of part (iii). Fig. 2(b) contains the derivative f' together with the bounds $-f^2/(1-F)$ and f^2/F as given in part (iv).