Journal of Statistical Planning and Inference 184 (2017) 18–24

Contents lists available at ScienceDirect

Journal of Statistical Planning and Inference

journal homepage: www.elsevier.com/locate/jspi

A theorem of Malliavin applied to the uniqueness of probabilistic moments

Xiangdong Yang

Department of Mathematics, Kunming University of Science and Technology, 650093 Kunming, China

ARTICLE INFO

Article history:
Received 7 March 2016
Received in revised form 26 October 2016
Accepted 7 November 2016
Available online 6 December 2016

MSC:
60E05
44A60

Keywords:
Determinate probability distributions
Moment problem
Completeness
Banach space
Malliavin's uniqueness theorem

ABSTRACT

The existing results on the moment problems are frequently characterized by Carleman criterion or Krein condition, which appeared a long time ago. Using the Malliavin's Uniqueness Theorem on analytic functions, we give new results in probabilistic moments problems. Our effort seems to be the first one in which the problem of vanishing moments is considered. Our approach is based on the completeness of function systems in Banach spaces. We prove that under mild restriction on the growth of the distribution functions, the moment problem is determinate as long as the lacunary λ_kth-order moments vanish. M-determinacy criterion is also proved wherever the distribution function is supported on a compact subset of the positive real axis.

1. Introduction

Let F be a distribution function (d.f.) of the random variable ξ and f be the corresponding density function of F whose support is $\mathbb{R} = (-\infty, +\infty)$. To simplify, all conditions and statements are expressed in terms of random variables (r.v.s) defined on an underlying probability space $(\Omega, \mathcal{F}, P)$ and their d.f.s and densities. Furthermore, suppose that the kth-order moments $\mathbb{E}(\xi^k) = M_k = \int_{-\infty}^{+\infty} x^k f(x)\mathrm{d}x$, $k = 1, 2, \ldots,$ are all finite. Then, F generates the *moment sequence* $\{M_k, k = 1, 2, \ldots, \}$ uniquely. The so-called moment problem (see Berg et al., 1979, Berg and Christensen, 1981, Berg, 1994, Berg, 1995, Lin, 1986, Lin, 1997, Lin and Stoyanov, 2014, Stoyanov, 2000 and Stoyanov et al., 2014) focuses on the cases where the moment sequence $\{M_k, k = 1, 2, \ldots, \}$ uniquely determines F. More accurately, let F and H be d.f.s with the distribution functions f and h, if $\int_{-\infty}^{+\infty} x^k f(x)\mathrm{d}x = \int_{-\infty}^{+\infty} x^k h(x)\mathrm{d}x$ for all $k = 1, 2, \ldots,$ imply $F = H$, we say F is *M-determinate*, otherwise, F is called *M-indeterminate*.

The *Carleman criterion* for the moment problem (see p. 126 in Koosis, 1988 where a comprehensive proof of Carleman criterion can be found by the approach of analysis, see also De Jeu, 2003) is as follows: if F is a d.f. on $(-\infty, +\infty)$ with all moments finite, then the condition $\Sigma_{k=1}^{\infty}(M_{2k})^{-1/2k} = \infty$ implies that F is M-determinate; if F is supported on $(0, +\infty)$, then $\Sigma_{k=1}^{\infty}(M_k)^{-1/2k} = \infty$ implies that F is M-determinate. In Stoyanov (2000) and Stoyanov et al. (2014), under the condition $M_{2k} > 0$ for all $k = 1, 2, \ldots,$, it is proved that the slow growth rate of the moments implies moment determinacy. The moment determinacy of random powers and products were also investigated.

E-mail address: yangsddp@126.com.

http://dx.doi.org/10.1016/j.jspi.2016.11.001
0378-3758/

The applications of polynomial decomposition and approximation results in the moment problem have a long history; among the existing results on this problem regarding moment determinacy, we mention the works (Bakan, 2001, 2002; Bakan and Ruscheweyh, 2005; Bakan, 2008; Hwang and Lin, 1984; Lin, 1986) for an overview. Using a conjugate argument of the Hahn–Banach Theorem in existence of the solution is a powerful tool. For the background of conjugate argument which is the main technique employed in this paper, we refer to Erdelyi (2005), Yang (2008) and Yang (2014). In this paper, we deal with the probabilistic moment problem to which uniqueness theorems for analytic functions can be applied, which is the main technique in the study of completeness of function systems in Banach spaces. By a *complete system* of elements $\{h_k\}$ of a Banach space B, we mean $\overline{Span}\{h_k\} = B$, i.e., the completeness is equivalent to the possibility of an arbitrary good approximation of any element of the space by linear combination of elements of this system. (See Erdelyi, 2005, Mergelyan, 1952, Mergelyan, 1953a, Mergelyan, 1953b, Yang, 2008 and Yang, 2014.) We will study the moments from a point view of completeness of functions in some Banach spaces.

The main purpose of the present paper is to introduce a vanishing moment problem for which random variables satisfy $\{M_{\lambda_k} = 0\}$ for a sequence of lacunary nonnegative increasing integers: $\{\lambda_k\} \subset \{1, 2 \dots, \}$ $k = 1, 2, \dots$. The motivation for this problem is the possibility to generalize the positive moment determinacy to the vanishing moment setting, and thus to apply the approximating method in a new way. We also investigate the determinacy of multivariate vanishing moment problem. The novelty of this paper lies in the application of the Malliavin's Uniqueness Theorem for analytic functions; it also lies in application of the completeness of function system to the study of moment determinacy.

This paper is divided into three sections. Section 2 is devoted to the study of the vanishing moment determinacy where the distribution function F is of one variable. We give the M-determinate criterions where the distribution function F is supported on the real axis $\mathbb{R}$ or on a compact set. In Section 3, we will study the vanishing moment determinacy where the distribution function F is of several variables.

2. Vanishing of the lacunary moments guarantees M-determinacy for one random variable

In this section, we will study the M-determinacy of a single random variable.

Fix a nonnegative continuous function $\alpha(t)$ which is defined on $\mathbb{R}$. Suppose that the density of F is $f(x)$ which satisfies $f(x) > 0$ and

$$f(t) \le \exp(-A\alpha(t)). \tag{1}$$

To present the main results, we need some background knowledge from complex analysis. From now on, the letter A will denote positive constants and it is not necessarily the same at each occurrence.

We will use Malliavin's uniqueness theorem for analytic functions (see Lemma 1 on p. 321 in Deng, 2000, and Malliavin, 1955) which is presented as follows:

Lemma 2.1. *Let $\gamma(t)$ be a nonnegative convex function on $\mathbb{R}$, denote by*

$$\gamma^*(t) = \sup\{xt - \gamma(t) : x \in \mathbb{R}\}.$$

Young transform of the function $\gamma(t)$. Suppose that $\Lambda(r)$ is an increasing function on $[0, \infty)$ satisfying

$$\Lambda(R) - \Lambda(r) \le A(\log R - \log r + 1)(R > r > 1).$$

Then, there exists a nonzero analytic function $G(z)$ in $\mathbb{C}_+$ satisfying

$$|G(z)| \le \exp\{Ax + \gamma(x) - x\Lambda(|z|)\}$$

if and only if

$$\int_0^{+\infty} \frac{\gamma^*(\{\Lambda(t) - a\})}{t^2 + 1} \mathrm{d}t < \infty \tag{2}$$

for some $a \in \mathbb{R}$.

We will make use of the following result on Fuchs product (see Lemma 9.5.9 on p. 159 of Boas, 1954).

Lemma 2.2. *Let $\Lambda = \{\lambda_k\}$ be a sequence of positive increasing numbers, then the function*

$$B(z) = \prod_{k=1}^{\infty} \frac{\lambda_k - z}{\lambda_k + z} \exp\left(\frac{2z}{\lambda_k}\right), \tag{3}$$

is analytic in the closed right half plane $\mathbb{C}_+$, and there exists a positive constant A such that

$$|B(z)| \le A \exp\{Ax + x\Lambda(|z|)\}, \quad z = x + iy \in \mathbb{C}_+ \tag{4}$$

$$|B(z)| \ge A \exp\{x\Lambda(|z|) - Ax\}, \quad z = x + iy \in \Delta(\Lambda) \tag{5}$$