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Tempered fractional Brownian and stable motions of second kind

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ABSTRACT

Meerschaert and Sabzikar (2013, 2016) introduced tempered fractional Brownian/stable motion (TFBM/TFSM) by including an exponential tempering factor in the moving average representation of FBM/FSM. The present paper discusses another tempered version of FBM/FSM, termed tempered fractional Brownian/stable motion of second kind (TFBM II/TFSM II). We prove that TFBM/TFSM and TFBM II/TFSM II are different processes. Particularly, large time properties of TFBM II/TFSM II are similar to those of FBM/FSM and are in deep contrast to large time properties of TFBM/TFSM.

1. Introduction

Meerschaert and Sabzikar (2016) introduced tempered fractional stable motion (TFSM) $Z_{H,\alpha,\lambda} = \{Z_{H,\alpha,\lambda}(t), t \in \mathbb{R}\}$ for $0 < \alpha \le 2, H > 0, \lambda > 0$ as stochastic integral

$$Z_{H,\alpha,\lambda}(t) := \int_{\mathbb{R}} \Big((t-y)_+^{H-\frac{1}{\alpha}} \mathrm{e}^{-\lambda(t-y)_+} - (-y)_+^{H-\frac{1}{\alpha}} \mathrm{e}^{-\lambda(-y)_+} \Big) M_\alpha(\mathrm{d}y), \tag{1.1}$$

with respect to α-stable Lévy process M_α. A particular case of TFSM termed the *tempered fractional Brownian motion* (TFBM) corresponding to $\alpha = 2$ and $M_2 = B$ (a standard Brownian motion) was studied in Meerschaert and Sabzikar (2013). Note that for $\lambda = 0$ (and $H \in (0,1)$) TFSM/TFBM agree with fractional stable/Brownian motion (FSM/FBM), see Samorodnitsky and Taqqu (1996). The role of the tempering by exponential factor in (1.1) manifests in the dependence properties of the increment process $Y_{H,\alpha,\lambda} = \{Y_{H,\alpha,\lambda}(t) := Z_{H,\alpha,\lambda}(t+1) - Z_{H,\alpha,\lambda}(t), t \in \mathbb{Z}\}$ called *tempered fractional stable noise* (TFSN) and *tempered fractional Gaussian noise* (TFGN) in the Gaussian case $\alpha = 2$. In particular, for small $\lambda > 0$ the autocovariance function of TFGN closely resembles that of fractional Gaussian noise (FGN) on an intermediate scale, but then it eventually falls off exponentially. On the other hand, the spectral density of TFGN vanishes at the origin for all $H > 0$ exhibiting a strong anti-persistent behavior, see Meerschaert and Sabzikar (2013).

In this paper we study a closely related but different tempered process called *tempered fractional Brownian/stable motion of second kind* (TFBM II/TFSM II) which is defined by replacing the integrand in (1.1) by

$$h_{H,\alpha,\lambda}(t;y) := (t-y)_+^{H-\frac{1}{\alpha}} \mathrm{e}^{-\lambda(t-y)_+} - (-y)_+^{H-\frac{1}{\alpha}} \mathrm{e}^{-\lambda(-y)_+} + \lambda \int_0^t (s-y)_+^{H-\frac{1}{\alpha}} \mathrm{e}^{-\lambda(s-y)_+}\, \mathrm{d}s, \qquad y \in \mathbb{R}. \tag{1.2}$$

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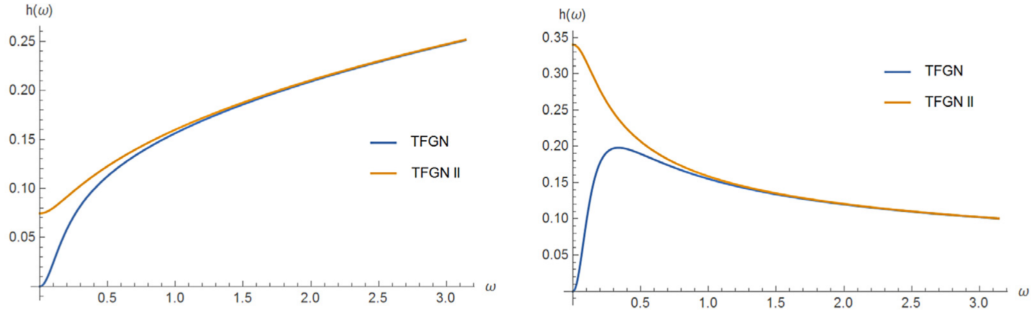

Fig. 1. Left: spectral density of TFGN and TFGN II with $H = 0.3$ and $\lambda = 0.15$. Right: the same plot for $H = 0.7$ and $\lambda = 0.15$.

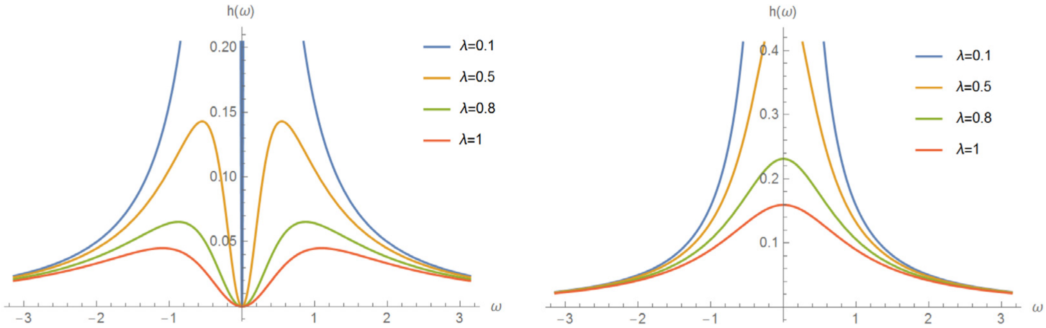

Fig. 2. Left: spectral density of TFGN with $H = 4/3$ and $\lambda = 0.1, 0.5, 0.8$, and 1. Right: spectral density of TFGN II for the same values of H and λ.

The corresponding α-stable process, denoted by $Z^{II}_{H,\alpha,\lambda} = \{Z^{II}_{H,\alpha,\lambda}(t), t \in \mathbb{R}\}$ is defined for all $H > 0, 1 < \alpha \leq 2, \lambda > 0$ and has stationary increments similarly as $Z_{H,\alpha,\lambda}$. (Although $Z^{II}_{H,\alpha,\lambda}$ can be defined for $\alpha \in (0, 1]$, see Remark 2.3, such an extension uses different (more complex) kernel from (1.2), and also seems to be less interesting from the point of view of applications. Therefore, we restrict ourselves to the case $\alpha \in (1, 2]$ for brevity of exposition.) The change of the integrand in the stochastic integral in (1.1) results in a drastic change of large-time behavior of the increment process

$$Y^{II}_{H,\alpha,\lambda} = \{Y^{II}_{H,\alpha,\lambda}(t) := Z^{II}_{H,\alpha,\lambda}(t+1) - Z^{II}_{H,\alpha,\lambda}(t), t \in \mathbb{Z}\} \tag{1.3}$$

called *tempered fractional Gaussian/stable noise of second kind* (TFGN II/TFSN II). The spectral density of TFGN II $Y^{II}_{H,2,\lambda}$ decays as a power function for frequencies $|\omega| > \lambda$ but remains bounded and separated from zero near zero frequency, making TFGN II a realistic model in turbulence and other applied areas. For example, in wind speed measurements, the spectral density follows this power law model for moderate frequencies, but the data deviates from that model at low frequencies, and the measured spectral density remains bounded (Pérez Beaupuits et al., 2004; Jang and Jyh-Shinn, 1999; Norton, 1981). Since the spectral density of TFGN II follows the same pattern, it can provide a useful model for such data. See also Figs. 1 and 2 and Remark 3.2.

One of the main motivation for our introducing and studying is the fact that these processes appear as the limits of the partial sums process of tempered stationary linear processes with discrete time and small tempering parameter $\lambda_N \sim \lambda/N$ tending to zero together with the sample size N. This problem is discussed in a parallel paper Sabzikar and Surgailis (2017) where we prove that the limit behavior of such partial process essentially depends on how fast the tempering parameter tends to zero, resulting in different limits in the strongly tempered ($\lim_{N\to\infty}\lambda_N/N = 0$), weakly tempered ($\lim_{N\to\infty}\lambda_N/N = \infty$), and moderately tempered ($\lim_{N\to\infty}\lambda_N/N \in (0, \infty)$) situations.

Let us describe the main results of this paper. Section 2 provides the basic definitions and properties of TFBM II/TFSM II. The latter include the spectral representation and the covariance function of TFBM II, relation to tempered fractional calculus (see Meerschaert and Sabzikar, 2014), and the relation between TFSM and TFSM II. Theorem 2.11 establishes local and global asymptotic self-similarity of TFSM and TFSM II. It shows that TFSM and TFSM II are very different processes; indeed, the former process is stochastically bounded and the latter process is stochastically unbounded as $t \to \infty$. Section 3 discusses the dependence properties of stationary processes TFSN II and TFGN II. We obtain the asymptotic behavior of the bivariate characteristic function of TFSN II which can be compared to the corresponding results for TFSN in Meerschaert and Sabzikar (2016) and for FSN in Astrauskas et al. (1991).