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On the expected diameter of planar Brownian motion

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ABSTRACT

Known results show that the diameter d_1 of the trace of planar Brownian motion run for unit time satisfies $1.595 \leq \mathbb{E}d_1 \leq 2.507$. This note improves these bounds to $1.601 \leq \mathbb{E}d_1 \leq 2.355$. Simulations suggest that $\mathbb{E}d_1 \approx 1.99$.

1. Introduction

Let $(b_t, t \in [0, 1])$ be standard planar Brownian motion, and consider the set $b[0, 1] = \{b_t : t \in [0, 1]\}$. The Brownian convex hull $\mathcal{H}_1 := \text{hull}\, b[0, 1]$ has been well-studied from Lévy (1948, §52.6, pp. 254–256) onwards; the expectations of the perimeter length ℓ_1 and area a_1 of $\mathcal{H}_1$ are given by the exact formulae $\mathbb{E}\ell_1 = \sqrt{8\pi}$ (due to Letac, 1978; Takács, 1980) and $\mathbb{E}a_1 = \pi/2$ (due to El Bachir, 1983).

Another characteristic is the *diameter*

$$d_1 := \text{diam}\, \mathcal{H}_1 = \text{diam}\, b[0, 1] = \sup_{x,y \in b[0,1]} \|x - y\|,$$

for which, in contrast, no explicit formula is known. The exact formulae for $\mathbb{E}\ell_1$ and $\mathbb{E}a_1$ rest on geometric integral formulae of Cauchy; since no such formula is available for d_1, it may not be possible to obtain an explicit formula for $\mathbb{E}d_1$. However, one may get bounds.

By convexity, we have the almost-sure inequalities $2 \leq \ell_1/d_1 \leq \pi$, the extrema being the line segment and shapes of constant width (such as the disc). In other words,

$$\frac{\ell_1}{\pi} \leq d_1 \leq \frac{\ell_1}{2}.$$

The formula of Letac (1978) and Takács (1980) says that $\mathbb{E}\ell_1 = \sqrt{8\pi}$, so we get:

Proposition 1. $\sqrt{8/\pi} \leq \mathbb{E}d_1 \leq \sqrt{2\pi}$.

Note that $\sqrt{8/\pi} \approx 1.5958$ and $\sqrt{2\pi} \approx 2.5066$. In this note we improve both of these bounds.

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2. Lower bound

For the lower bound, we note that $b[0, 1]$ is compact and thus, as a corollary of Lemma 6, we have the formula

$$d_1 = \sup_{0 \le \theta \le \pi} r(\theta), \tag{1}$$

where r is the parametrized range function given by

$$r(\theta) = \sup_{0 \le s \le 1} (b_s \cdot \mathbf{e}_\theta) - \inf_{0 \le s \le 1} (b_s \cdot \mathbf{e}_\theta),$$

with $\mathbf{e}_\theta$ being the unit vector $(\cos\theta, \sin\theta)$. Feller (1951) established that

$$\mathbb{E}r(\theta) = \sqrt{8/\pi} \quad \text{and} \quad \mathbb{E}(r(\theta)^2) = 4\log 2, \tag{2}$$

and the density of $r(\theta)$ is given explicitly as

$$f(r) = \frac{8}{\sqrt{2\pi}} \sum_{k=1}^{\infty} (-1)^{k-1} k^2 \exp\{-k^2 r^2/2\}, \ (r \ge 0). \tag{3}$$

Combining (1) with (2) gives immediately $\mathbb{E}d_1 \ge \mathbb{E}r(0) = \sqrt{8/\pi}$, which is just the lower bound in Proposition 1. For a better result, a consequence of (1) is that $d_1 \ge \max\{r(0), r(\pi/2)\}$. Observing that $r(0)$ and $r(\pi/2)$ are independent, we get:

Lemma 2. *$\mathbb{E}d_1 \ge \mathbb{E}\max\{X_1, X_2\}$, where X_1 and X_2 are independent copies of $X := r(0)$.*

It seems hard to explicitly compute $\mathbb{E}\max\{X_1, X_2\}$ in Lemma 2, because although the density given at (3) is known explicitly, it is not very tractable. Instead we obtain a lower bound. Since

$$\max\{x, y\} = \frac{1}{2}(x + y + |x - y|)$$

we get

$$\mathbb{E}\max\{X_1, X_2\} = \mathbb{E}X + \frac{1}{2}\mathbb{E}|X_1 - X_2|. \tag{4}$$

Thus with Lemma 2, the lower bound in Proposition 1 is improved given any non-trivial lower bound for $\mathbb{E}|X_1 - X_2|$. Using the fact that for any $c \in \mathbb{R}$, if m is a median of X, $\mathbb{E}|X - c| \ge \mathbb{E}|X - m|$, we see that

$$\mathbb{E}|X_1 - X_2| \ge \mathbb{E}|X - m|.$$

Again, the intractability of the density at (3) makes it hard to exploit this. Instead, we provide the following as a crude lower bound on $\mathbb{E}|X_1 - X_2|$.

Lemma 3. *For any $a, h > 0$,*

$$\mathbb{E}|X_1 - X_2| \ge 2h\,\mathbb{P}(X \le a)\,\mathbb{P}(X \ge a + h).$$

Proof. We have

$$\begin{aligned}
\mathbb{E}|X_1 - X_2| &\ge \mathbb{E}\left[|X_1 - X_2|\mathbf{1}\{X_1 \le a, X_2 \ge a + h\}\right] + \mathbb{E}\left[|X_1 - X_2|\mathbf{1}\{X_2 \le a, X_1 \ge a + h\}\right] \\
&\ge h\,\mathbb{P}(X_1 \le a)\,\mathbb{P}(X_2 \ge a + h) + h\,\mathbb{P}(X_2 \le a)\,\mathbb{P}(X_1 \ge a + h) \\
&= 2h\,\mathbb{P}(X \le a)\,\mathbb{P}(X \ge a + h)
\end{aligned}$$

which proves the statement. □

This lower bound yields the following result.

Proposition 4. *For $a, h > 0$ define*

$$g(a, h) := h\left(\frac{4}{\pi}\exp\left\{-\frac{\pi^2}{2a^2}\right\} - \frac{4}{3\pi}\exp\left\{-\frac{9\pi^2}{2a^2}\right\}\right)\left(1 - \frac{4}{\pi}\exp\left\{-\frac{\pi^2}{8(a+h)^2}\right\}\right).$$

Then $\mathbb{E}d_1 \ge \sqrt{8/\pi} + g(1.492, 0.337) \approx 1.6014$.

Proof. Consider

$$Z := \sup_{0 \le s \le 1} |b_s \cdot \mathbf{e}_0|.$$