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The randomly fluctuating hyperrectangles are spatially monotone

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We show that the probability of a site being occupied at any instance of time in the one-dimensional randomly fluctuating hyperrectangles processes decreases monotonically with respect to its distance from the origin.

1. Introduction

In this short note, we introduce a new class of Markovian spatial growth processes, which we call the randomly fluctuating hyperrectangles. We are interested in the property of the state of the process being at all times more likely to contain points that are nearer to the origin rather than points that are farther away. In a spatial stochastic process, a site is said to be occupied (or not) at a certain time according to whether (or not) it is included at the state of the process at that time. Properties regarding occupied site probabilities are of inherent interest in the study of spatial stochastic processes, since these probabilities play a key rôle in their analysis and understanding, in addition to that it is natural to expect for this same reason their involvement in applications. Whereas occupied site probabilities are monotonically decreasing functions of their spatial coordinate at every fixed instance of time is interesting in its own right problem that has been studied for various spatial stochastic processes, to which we refer to after mentioning our result below.

We may briefly define the class of randomly fluctuating hyperrectangles processes informally as follows. The state-space of the process comprises of every hyperrectangle with vertices having integer-valued coordinates. The process evolves in discrete-time according to contraction phases alternating with expansion phases. In the standard specifications case, contractions comprise of considering the sub-hyperrectangle obtained by sampling uniformly at random among all sub-hyperrectangles of the current hyperrectangle; whereas expansions comprise of shifting every face of the current hyperrectangle vertically in increasing direction according to independent geometrically distributed random values, and considering the sup-hyperrectangle formed by appropriately extending each shifted face. For instance, in two spatial dimensions an explicit definition of the process would be as follows. Let $\mathbf{R}$ be the set of (finite) rectangles with integer-coordinate vertices that, without loss of generality due to rotational invariance of the dynamics below, we assume to have sides parallel to the coordinate axes. Further, let $N(\zeta)$, $\zeta \in \mathbf{R}$, denote the coordinate value that points of the north side of ζ have in common, i.e. their projection on the vertical coordinate axis, and define $S(\zeta)$, $E(\zeta)$, $W(\zeta)$ analogously with regard to its south, east and west sides, respectively. Let also $\mathbf{R}(\zeta) = \{\xi \in \mathbf{R} : \xi \subseteq \zeta\}$, i.e. the σ-algebra of subsets of ζ in $\mathbf{R}$. Let $(X^i(0), X^i(1), \ldots)$, $i = N, S, E, W$ be independent collections of i.i.d. random variables such that $\mathbb{P}(X^i(t) = n) = (1-p)p^{n-1}$,

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$n = 1, 2, \ldots$. The standard specifications randomly fluctuating rectangles process $(\zeta_t : t \geq 0)$ with parameter p is a discrete-time Markov process on $\mathbf{R} \cup \emptyset$ the transition rates of which are determined as follows: given ζ_t, we sample $\tilde{\zeta}_t$ uniformly at random[1] from $\mathbf{R}(\zeta_t) \cup \emptyset$ and, whenever $\tilde{\zeta}_t \neq \emptyset$, let ζ_{t+1} be such that $i(\zeta_{t+1}) = i(\tilde{\zeta}_t) + X_t^i$ for $i = N, E$, and that $i(\zeta_{t+1}) = i(\tilde{\zeta}_t) - X_t^i$ for $i = S, W$, otherwise let $\zeta_k = \emptyset$ for all $k \geq t + 1$. Thus, contractions of this type comprise of sampling a sub-rectangle according to the probability measure which assigns equal mass to all elements, including the empty set; whereas expansions will always comprise of independent geometrically distributed outward-shifting of each side, and considering the sup-rectangle formed by joining them.

We may briefly summarize some of the intrinsic features exhibited by the randomly fluctuating hyperrectangles processes as follows. We note that the process in the standard specifications case lacks an obvious graphical representation, i.e. a coupling construction of versions of the process with different starting states, and hence does not fall into the general framework of interacting particle systems, or percolation processes. Further, regarding the restriction to finite initial states, we note that the uniform contraction rule is not otherwise well-defined, as an aftereffect of the mere fact that uniform (i.e. assigning equal probability to every element) probability measures on countable spaces are not compatible with the standard axioms of probability theory. We also note that it is not difficult to see that this class of Markov processes is irreducible, in that, for any state ζ, there is t such that $\mathbb{P}(\zeta_t = \zeta) > 0$.

Our Theorem 1 stated in the next section regards the randomly fluctuating hyperrectangles class of processes in one (spatial) dimension started from the single site at the origin. Although we work out the details in our proofs in the one-dimensional case only, we nevertheless find that our result and arguments extend analogously in any dimension. Further, whereas the specification of a geometric distribution for the expansion phases is essential here, the result and technique of proof apply directly for the randomly fluctuating hyperrectangles processes with other, qualitatively different than uniform types of contraction specifications (see Remarks 1 and 2). We show in Theorem 1 that at any instance of time the probability that a point is occupied decreases monotonically with respect to its distance from the origin. We furthermore show in Theorem 1 that, at any fixed instance of time, the occupied site probability is an even function; that is, that the probability that a site is occupied is equal to the probability that its symmetric about the origin counterpart site is occupied.

We refer to studies regarding the corresponding spatial monotonicity property for other stochastic spatial growth processes as follows. In regard to the basic one-dimensional contact process, which is the continuous-time analog of two-dimensional oriented percolation, Gray (1991) introduced the spatial monotonicity of its occupied site probabilities, among other intriguing properties regarding them. The detailed and elaborate proof of this notable result is given by Andjel and Gray (2016) and by Andjel and Sued (2008). Whereas the corresponding property holds for undirected percolation on integer lattices is in general an open question. A partial result in the direction of a positive reply is obtained by De Lima et al. (2015). Whereas occupied site probabilities are monotone for general one-dimensional attractive spin-systems is also in the case of finite initial configurations an important open problem. Regarding first-passage percolation, Hammersley and Welsh (1965) raised the corresponding question of spatial monotonicity for first-passage times, which also remains to date an open question. For a partial result in the direction of a negative reply, see van den Berg (1983); for partial results in the direction of a positive reply, see the more recent work by Gouéré (2014) and the references therein. We also note that the corresponding stochastic monotonicity result regarding symmetric branching random walks is derived in Lemma 11 by Lalley and Zheng (2011), the proof of which relies crucially on the independence of the descendancy of distinct particles, due to permitting an arbitrary number of particles per site in this process.

The remainder of this note is organized as follows. We state our Theorem 1 in Section 2. We give its proof in Section 3.

2. Statement of Theorem 1

The randomly fluctuating intervals process is defined as follows. Let $\mathbf{I}(\zeta)$ be the set of all integer interval subsets of $\zeta \subseteq \mathbb{Z}$ including the empty set, where $\mathbb{Z}$ denotes the integers, and simply write $\mathbf{I}$ for $\mathbf{I}(\mathbb{Z})$. Let further $R(\zeta) = \sup \zeta$ and $L(\zeta) = \inf \zeta$, with the convention that $\sup \emptyset = -\infty$. Furthermore, throughout here, $(N_t^L : t \geq 0)$ and $(N_t^R : t \geq 0)$ will denote independent collections of i.i.d. geometric r.v. such that $\mathbb{P}(N_t^L = n) = (1-p)p^{n-1}$, $n = 1, 2, \ldots$, where $p \in (0, 1)$ is called the expansion parameter. The standard specifications randomly fluctuating intervals are Markov processes $(\zeta_t : t \geq 0)$ on $\mathbf{I}$ with parameter $p \in (0, 1)$ defined as follows: Given ζ_t, choose $\tilde{\zeta}_t$ uniformly at random from $\mathbf{I}(\zeta_t)$ and, whenever $\tilde{\zeta}_t \neq \emptyset$, set $\zeta_{t+1} = \{L(\tilde{\zeta}_t) - N_t^L, \ldots, R(\tilde{\zeta}_t) + N_t^R\}$, otherwise set $\zeta_k = \emptyset$ for all $k \geq t + 1$. To state our theorem next, let ζ_t^O be the standard specifications randomly fluctuating intervals process that started at the origin, and let also $f_t(x) = \mathbb{P}(x \in \zeta_t^O)$.

Theorem 1. *For all* t, $f_t(\mathrm{x})$ *is an even function that is decreasing in* $|x|$.

Remark 1. The arguments in the proof can be adapted to also establish this result for the randomly fluctuating intervals with general contraction. We describe this Markov process (η_t) on $\mathbf{I}$. Given η_t, sample $\tilde{\eta}_t$ uniformly at random among intervals in $\mathbf{I}(\eta_t)$ with size $X \sim \phi(k; |\eta_t|)$, where $\phi(k; n)$ is an arbitrary probability mass function of a discrete random variable assuming values $k = 0, \ldots, n$ and, whenever $\tilde{\eta}_t \neq \emptyset$, set $\eta_{t+1} = \{L(\tilde{\eta}_t) - N_t^L, \ldots, R(\tilde{\eta}_t) + N_t^R\}$, otherwise set $\eta_k = \emptyset$ for all $k \geq t + 1$.

[1] That is, according to the uniform probability measure, $\mathbb{P}(\tilde{\zeta}_t = \zeta \,|\, \zeta_t = \eta) = \frac{1}{|\mathbf{R}(\eta)+1|}$, for all $\zeta \in \mathbf{R}(\eta) \cup \emptyset$.