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Moment conditions in strong laws of large numbers for multiple sums and random measures[☆]

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ABSTRACT

The validity of the strong law of large numbers for multiple sums $S_{\boldsymbol{n}}$ of independent identically distributed random variables $Z_{\boldsymbol{k}}$, $\boldsymbol{k} \le \boldsymbol{n}$, with r-dimensional indices is equivalent to the integrability of $|Z|(\log^+|Z|)^{r-1}$, where Z is the generic summand. We consider the strong law of large numbers for more general normalizations, without assuming that the summands $Z_{\boldsymbol{k}}$ are identically distributed, and prove a multiple sum generalization of the Brunk–Prohorov strong law of large numbers. In the case of identical finite moments of order $2q$ with integer $q \ge 1$, we show that the strong law of large numbers holds with the normalization $(n_1 \cdots n_r)^{1/2}(\log n_1 \cdots \log n_r)^{1/(2q)+\varepsilon}$ for any $\varepsilon > 0$.

The obtained results are also formulated in the setting of ergodic theorems for random measures, in particular those generated by marked point processes.

1. Introduction

Let $r \ge 1$ be an integer number and let $\mathbb{N}^r$ denote the set of r-dimensional vectors with positive integer coordinates. Elements of $\mathbb{N}^r$ are denoted by $\boldsymbol{k}$, $\boldsymbol{n}$, etc. The inequality $\boldsymbol{k} \le \boldsymbol{n}$ is defined coordinatewisely, that is $k_i \le n_i$, $1 \le i \le r$, where $\boldsymbol{k} = (k_1, \dots, k_r)$ and $\boldsymbol{n} = (n_1, \dots, n_r)$. Denote $|\boldsymbol{n}| = n_1 \cdots n_r$. Then, $|\boldsymbol{n}| \to \infty$ means that the maximum of all coordinates of $\boldsymbol{n}$ converges to infinity and so is called max-convergence or product convergence, see Klesov (2014). Furthermore, $\boldsymbol{n} \to \infty$ means that all components of $\boldsymbol{n}$ converge to infinity, that is $\min(n_1, \dots, n_r) \to \infty$, it is called the min-convergence in Klesov (2014).

Consider an array $\{b_{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N}^r\}$ of positive numbers indexed by $\mathbb{N}^r$ such that $b_{\boldsymbol{n}} \to \infty$ as $|\boldsymbol{n}| \to \infty$. Define partial sums of random variables $\{Z_{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N}^r\}$ by

$$S_{\boldsymbol{n}} = \sum_{\boldsymbol{k} \le \boldsymbol{n}} Z_{\boldsymbol{k}}, \quad \boldsymbol{n} \in \mathbb{N}^r.$$

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The random field $\{Z_{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N}^r\}$ is said to satisfy the strong law of large numbers with the normalization $\{b_{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N}^r\}$ if $Z_{\boldsymbol{n}}$ is integrable for all $\boldsymbol{n}$ and

$$\frac{1}{b_{\boldsymbol{n}}}(S_{\boldsymbol{n}} - \mathbf{E}S_{\boldsymbol{n}}) \to 0 \qquad \text{a.s. as } |\boldsymbol{n}| \to \infty. \tag{1}$$

If all $Z_{\boldsymbol{n}}$'s are centered or are not integrable, the validity of the strong law of large numbers means that

$$\frac{1}{b_{\boldsymbol{n}}}S_{\boldsymbol{n}} \to 0 \qquad \text{a.s. as } |\boldsymbol{n}| \to \infty. \tag{2}$$

It is easy to see that $b_{\boldsymbol{n}}$ should grow faster than $\sqrt{|\boldsymbol{n}|}$. If $\{Z_{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N}^r\}$ are independent copies of a centered random variable Z, then (2) for $b_{\boldsymbol{n}} = |\boldsymbol{n}|$ becomes the strong law of large numbers for multiple sums, which holds if and only if $\mathbf{E}[|Z|(\log^+|Z|)^{r-1}] < \infty$, see Smythe (1973). Here $\log^+ t$ denotes the positive part of $\log t$. If $b_{\boldsymbol{n}}$ grows faster than $|\boldsymbol{n}|$, the corresponding results are variants of the Marcinkiewicz–Zygmund law. In this paper we present a whole spectrum of such results exploring relations between the strength of the moment conditions and the growth rate of the sequence of normalizing constants. In particular, we show that imposing sufficiently strong moment assumptions makes it possible to bring the normalizing factors to $b_{\boldsymbol{n}} = |\boldsymbol{n}|^{1/2}(\log n_1 \cdots \log n_r)^{\varepsilon}$ for any $\varepsilon > 0$.

The strong law of large numbers was used in Smythe (1975) to derive the ergodic theorem for sums generated by marked point processes. We first provide an alternative proof (that gives a stronger result under weaker conditions) of the strong law of large numbers claimed in Smythe (1975) to follow from the multivariate analogue of the Kronecker lemma. As we show in Section 4, this lemma holds only in the nonnegative case. Indeed, we provide a counterexample to a "natural" generalization of the Kronecker lemma which invalidates the proof of Smythe (1975, Th. 2.1.1).

Section 2 contains several strong laws of large numbers for multiple sums of not identically distributed random variables that combine moment conditions on the summands with not so fast growing normalizing constants. Along the same line, we generalize the Brunk–Prohorov criterion for the validity of the strong law of large numbers known for the case of univariate sums, see Brunk (1948) and Prohorov (1950). In case of i.i.d. summands, the conditions simplify substantially.

Section 3 rephrases the results from Section 2 for random measures, in particularly, those generated by marked point processes.

2. Strong laws of large numbers for multiple sums

2.1. Conditions on moments of order up to 2

The field $\{b_{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N}^r\}$ is said to be monotonic if $b_{\boldsymbol{k}} \le b_{\boldsymbol{n}}$ for $\boldsymbol{k} \le \boldsymbol{n}$ coordinatewisely. Define the increments of $\{b_{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N}^r\}$ by

$$\Delta[b_{\boldsymbol{n}}] = \sum_{\boldsymbol{m}=(m_1,\dots,m_r)\in\{0,1\}^r} (-1)^{m_1+\cdots+m_r} \boldsymbol{b}_{\boldsymbol{n}-\boldsymbol{m}},$$

where the array $\{b_{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N}^r\}$ is extended for indices with *non-negative* coordinates by letting $b_{\boldsymbol{n}} = 0$ if at least one of the coordinates of $\boldsymbol{n}$ vanishes. The non-negativity of $\Delta[b_{\boldsymbol{n}}]$ for all $\boldsymbol{n}$ is a stronger condition than the monotonicity of $\{b_{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N}^r\}$.

Theorem 2.1 appears as Smythe (1975, Th. 2.1.1) and was announced first in Smythe (1973). However, it was formulated in the particular case $\boldsymbol{n} \to \infty$ and assuming the non-negativity of increments $\Delta[b_{\boldsymbol{n}}]$ for the weights. In order to deduce the strong law of large numbers from the convergence of random multiple series, it relied on the Kronecker lemma for multiple sums that was mentioned as a "simple generalization" of the univariate case in Smythe (1975, p. 116). It will be explained in Section 4 that such a generalization holds only assuming that the summands are non-negative, and so the proof of Smythe (1975, Th. 2.1.1) was not complete. We suggest an alternative proof that derives the strong law of large numbers under the max-convergence $|\boldsymbol{n}| \to \infty$, and for this it is unavoidable to assume that

$$b_{\boldsymbol{n}} \to \infty \text{ as } |\boldsymbol{n}| \to \infty \tag{3}$$

instead of $\boldsymbol{n} \to \infty$ in Smythe (1975). The one-dimensional case is considered in Fazekas and Klesov (2000).

Note that the convergence of multiple series $\sum_{\boldsymbol{n}\in\mathbb{N}^r} a_{\boldsymbol{n}}$ is always understood as the convergence of their partial sums $\sum_{\boldsymbol{k}\le\boldsymbol{n}} a_{\boldsymbol{k}}$ as $\boldsymbol{n} \to \infty$.

Theorem 2.1. *Assume that* $\{b_{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N}^r\}$ *is monotonic. Let* φ *be a positive even continuous function on* $\mathbb{R}$ *such that* $x^{-1}\varphi(x)$ *is non-decreasing and* $x^{-2}\varphi(x)$ *is non-increasing for* $x > 0$. *If* $\{Z_{\boldsymbol{n}}, \boldsymbol{n} \in \mathbb{N}^r\}$ *are independent centered random variables such that*

$$\sum_{\boldsymbol{k}\in\mathbb{N}^r} \frac{\mathbf{E}\varphi(Z_{\boldsymbol{k}})}{\varphi(b_{\boldsymbol{k}})} < \infty, \tag{4}$$

then the series $\sum_{\boldsymbol{k}\in\mathbb{N}^r} Z_{\boldsymbol{k}}/b_{\boldsymbol{k}}$ *converges almost surely and* (1) *holds.*