Statistics and Probability Letters 131 (2017) 78–86

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Piecewise linear process with renewal starting points

Nikita Ratanov

Universidad del Rosario, Cl. 12c, No. 4-69, Bogotá, Colombia

ARTICLE INFO

Article history:
Received 9 June 2017
Received in revised form 19 August 2017
Accepted 21 August 2017
Available online 30 August 2017

MSC:
primary 60J27
secondary 60J75
60K99

Keywords:
Continuous-time homogeneous Markov chain
Piecewise linear process
Renewal process
First passage time

ABSTRACT

This paper concerns a Markovian piecewise linear process, based on a continuous-time Markov chain with a finite state space. The process describes the movement of a particle that takes a new linear trend starting from a new random point (with state-dependent distribution) after each trend switch. The distribution of particle's position is derived in a closed form. In some special cases the distributions of the level passage times are provided explicitly.

1. Introduction and main definitions

Let $\varepsilon = \varepsilon(t), \quad t \geq 0$, be a continuous-time homogeneous Markov chain with the finite state space E. Let $\Phi(t) = \mathrm{e}^{t\Lambda} = \left(\Phi_{ij}(t)\right)_{i,j\in E}$ be the transition semi-group with the infinitesimal generator Λ. Consider the sequence of switching times, $0 < T_1 < T_2 < \cdots < T_n < \cdots, \quad T_0 = 0$, and denote by $N(t)$ the number of switchings occurred up to time t, $N(t) = \max\{n \mid T_n \leq t\}$.

We study the piecewise linear random process

$$X(t) = x_{N(t)} + c_{\varepsilon(T_{N(t)})}(t - T_{N(t)}), \qquad t \geq 0. \tag{1.1}$$

Here $c_i, \ i \in E$, are deterministic constants, and $x_n, \ n \geq 0$, are independent random variables with distributions determined by the current state $\varepsilon(T_n)$.

Process $X(t), \ t \geq 0$, describes the location of a particle that moves linearly and takes a new starting point after each trend switch. Process X resembles well-studied Markovian growth-collapse processes, see e.g. Boxma et al. (2006). In contrast with (1.1) the growth-collapse processes presume a constant trend with additive or multiplicative *down* jumps. Such models occur in insurance mathematics and related fields, see Asmussen (2003, XIV-5) or Rolski et al. (1999, Chapters 5 and 11), and in production/inventory models studied by Shanthikumar and Sumita (1983), among others.

On the other hand, (1.1) is related to jump-telegraph process $\mathbb{T}(t) := \int_0^t c_{\varepsilon(u)}\mathrm{d}u + \int_0^t Y_{\varepsilon(u-)}\mathrm{d}N_u$, which is a piecewise linear process jumping from the current position, see e.g. Di Crescenzo et al. (2013) and López and Ratanov (2012); piecewise linear processes with jumps are presented by Ratanov (2014). Jump-telegraph processes are widely applied in various fields, including financial market modelling, see Kolesnik and Ratanov (2013).

E-mail address: nikita.ratanov@urosario.edu.co.

http://dx.doi.org/10.1016/j.spl.2017.08.010
0167-7152/

In contrast to jump-telegraph process, $X(t),\ t \geq 0$, (1.1), completely updates starting points. Possible applications of this type of processes could be in financial modelling and insurance mathematics.

The paper is structured as follows. In Section 2 we give explicit formulae for transition densities, expectations and limit distributions of $X(t)$ (as $t \to \infty$). Section 3 is devoted to a detailed analysis of the level passage times in the case of the two-state underlying Markov process ε.

2. Distribution

Define the transition probabilities $\mathbf{p}(t, \mathrm{d}y \mid x)$ and $\mathbf{p}(t, \mathrm{d}y)$ by entries

$$p_i(t, \mathrm{d}y \mid x) = \mathbb{P}\{X(t) \in \mathrm{d}y \mid \varepsilon(0) = i,\ X(0) = x\}, \qquad -\infty < x, y < \infty, \tag{2.1}$$

and

$$p_i(t, \mathrm{d}y) = \int_{-\infty}^{\infty} p_i(t, \mathrm{d}y \mid x) g_i(\mathrm{d}x), \qquad -\infty < y < \infty, \quad i \in E, \tag{2.2}$$

where $g_i(\mathrm{d}x)$ is the distribution of the initial starting point at the initial state $i = \varepsilon(0)$. By conditioning on the first switching one can obtain the following integral equations

$$p_i(t, \mathrm{d}y \mid x) = \mathrm{e}^{-\lambda_i t}\delta_{x+c_i t}(\mathrm{d}y) + \sum_{j\in E,\ j\neq i} \int_0^t \lambda_{ij}\mathrm{e}^{-\lambda_i \tau} p_j(t-\tau, \mathrm{d}y)\mathrm{d}\tau, \qquad i \in E, \tag{2.3}$$

where $\delta_a(\mathrm{d}y)$ denotes δ-measure localised at point a and $\lambda_i = \sum_{j\in E,\ j\neq i}\lambda_{ij}$. Further, by (2.2)

$$p_i(t, \mathrm{d}y) = \mathrm{e}^{-\lambda_i t} g_i^{tc_i}(\mathrm{d}y) + \sum_{j\in E,\ j\neq i} \int_0^t \lambda_{ij}\mathrm{e}^{-\lambda_i \tau} p_j(t-\tau, \mathrm{d}y)\mathrm{d}\tau, \qquad i \in E. \tag{2.4}$$

Here $g_i^a(\mathrm{d}y)$ is the displacement of measure g_i: for any integrable test-function ϕ

$$\int_{-\infty}^{\infty} \phi(y) g_i^a(\mathrm{d}y) = \int_{-\infty}^{\infty} \phi(y+a) g_i(\mathrm{d}y).$$

Systems (2.3) and (2.4) can be solved explicitly. Note that when all trends vanish, $c_i = 0,\ i \in E$, the distribution of $X(t) = x_{N(t)}$ is given by $\mathbb{P}\{X(t) \in \mathrm{d}x \mid \varepsilon(0) = i\} = \sum_{j\in E} \Phi_{ij}(t) g_j(\mathrm{d}x)$. where $\Phi_{ij}(t)$ are the entries of the transition semi-group $\mathrm{e}^{t\Lambda}$ introduced in Section 1.

Theorem 2.1. *The transition probabilities* $\mathbf{p}(t, \mathrm{d}y \mid x)$ *and* $\mathbf{p}(t, \mathrm{d}y)$, (2.1)–(2.2), *have the form*

$$p_i(t, \mathrm{d}y \mid x) = \mathrm{e}^{-\lambda_i t}\delta_{x+tc_i}(\mathrm{d}y) + \sum_{j\in E} \int_0^t \Phi_{ij}(t-\tau) \left[\sum_{k\in E\ k\neq j} \lambda_{jk}\mathrm{e}^{-\lambda_k \tau} g_k^{\tau c_k}(\mathrm{d}y) \right] \mathrm{d}\tau, \tag{2.5}$$

$$p_i(t, \mathrm{d}y) = \mathrm{e}^{-\lambda_i t} g_i^{tc_i}(\mathrm{d}y) + \sum_{j\in E} \int_0^t \Phi_{ij}(t-\tau) \left[\sum_{k\in E\ k\neq j} \lambda_{jk}\mathrm{e}^{-\lambda_k \tau} g_k^{\tau c_k}(\mathrm{d}y) \right] \mathrm{d}\tau, \quad i \in E. \tag{2.6}$$

Let

$$M_i(t) = \int_{-\infty}^{\infty} \mathbb{E}\{X_i(t) \mid \varepsilon(0) = i,\ X_i(0) = x\} g_i(\mathrm{d}x), \qquad i \in E.$$

Then,

$$M_i(t) = \mathrm{e}^{-\lambda_i t}(m_i + tc_i) + \sum_{j\in E} \int_0^t \Phi_{ij}(t-\tau) \left[\sum_{k\in E,\ k\neq j} \lambda_{jk}\mathrm{e}^{-\lambda_k \tau}(m_k + \tau c_k) \right] \mathrm{d}\tau, \tag{2.7}$$

where $m_i = \int_{-\infty}^{\infty} x g_i(\mathrm{d}x),\ i \in E$.

Proof. By conditioning on the last switching time and using the time-reversal property, see e.g. Brémaud (1999), one can derive (2.5): the first term corresponds to the case of no switchings and other summands describe the movement of the particle, which starts at time 0 from the state $i \in E$ and makes the last switching at time $t - \tau$. Eq. (2.6) follows from (2.2). Formula (2.7) follows from (2.6). □