Statistics and Probability Letters 131 (2017) 93–101

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Sharp weighted weak-norm estimates for maximal functions

Michał Brzozowski, Adam Osękowski *, Mateusz Rapicki

Department of Mathematics, Informatics and Mechanics, University of Warsaw, Banacha 2, 02-097 Warsaw, Poland

ARTICLE INFO

Article history:
Received 28 May 2017
Received in revised form 13 August 2017
Accepted 21 August 2017
Available online 31 August 2017

MSC 2010:
primary 42B25
secondary 60G44

Keywords:
Maximal
Martingale
Bellman function
Best constants

ABSTRACT

For any $1 < p < \infty$ and any $c \geq 1$ we identify the least constant $C_{p,c}$ with the following property. If $X = (X_t)_{t\geq 0}$ is a uniformly integrable martingale and $W = (W_t)_{t\geq 0}$ is a weight satisfying Muckenhoupt's condition A_p with $[W]_{A_p} \leq c$, then we have the Lorentz-norm estimate

$$\left\| \sup_{t\geq 0} |X_t| \right\|_{L^{p,\infty}(W)} \leq C_{p,c} \|X_\infty\|_{L^{p,\infty}(W)}.$$

The proof exploits related sharp weak-type estimates and optimization arguments.

1. Introduction

The paper is devoted to the study of sharp weighted versions of the classical maximal estimates for real-valued martingales obtained by Doob (1990). Let us start with the necessary background, notation and the statement of related results. Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, filtered by $(\mathcal{F}_t)_{t\geq 0}$, a nondecreasing family of sub-σ-fields of $\mathcal{F}$, such that $\mathcal{F}_0$ contains all the events of probability 0. Let X be an adapted, real-valued, uniformly integrable martingale with right-continuous trajectories that have limits from the left; such a martingale converges almost surely to an integrable variable which will be denoted by X_∞. The maximal function of X is given by $X^* = \sup_{s\geq 0} |X_s|$ and the square bracket of X is denoted by $[X, X]$ (see e.g. Dellacherie and Meyer, 1982 for the definition). A classical result of Doob (1990) asserts that the maximal function satisfies the weak-type (p, p) estimate

$$\|X^*\|_{L^{p,\infty}} \leq \|X_\infty\|_{L^p}, \qquad 1 \leq p < \infty,$$

where $\|X^*\|_{L^{p,\infty}} = \sup_{\lambda>0} [\lambda^p \mathbb{P}(X^* \geq \lambda)]^{1/p}$ is the usual weak pth norm of X^*. Furthermore, if $1 < p \leq \infty$, then we have the strong-type bound

$$\|X^*\|_{L^p} \leq \frac{p}{p-1} \|X_\infty\|_{L^p}.$$

Both estimates above are sharp: for any value of p, the constants 1 and $p/(p-1)$ cannot be improved. There is a related result proved by Osękowski in Osękowski (2015) which provides a sharp comparison of weak pth norms of X and X^*: for

* Corresponding author.
E-mail addresses: M.Brzozowski@mimuw.edu.pl (M. Brzozowski), A.Osekowski@mimuw.edu.pl (A. Osękowski), M.Rapicki@mimuw.edu.pl (M. Rapicki).

http://dx.doi.org/10.1016/j.spl.2017.08.011
0167-7152/

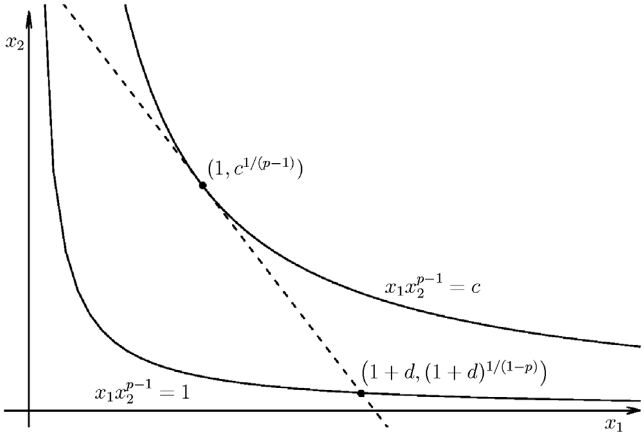

Fig. 1. The geometric interpretation of the number $d = d(p, c)$.

any $1 < p < \infty$ we have

$$\|X^*\|_{L^{p,\infty}} \leq \frac{p}{p-1}\|X_\infty\|_{L^{p,\infty}}, \tag{1.1}$$

and the constant $p/(p-1)$ is again the best possible. See also Osękowski (2014) for a related sharp $L^{q,\infty} \to L^p$ bound.

The primary goal of this paper is to study weighted version of the estimate (1.1). Here the word 'weight' will refer to a positive, uniformly integrable martingale $W = (W_t)_{t\geq 0}$. Such a process gives rise to a new (not necessarily probability) measure $W_\infty \mathrm{d}\mathbb{P}$. For technical reasons, we will also assume that W has continuous paths with probability 1. It is well-known that without any regularity assumptions on the trajectories of the weight almost all reasonable inequalities fail to hold (cf. the paper (Izumisawa et al., 1979) for a related fact for BMO martingales).

When studying weighted L^p or weak-L^p estimates for maximal functions, one has to restrict oneself to the so-called A_p weights. Let us discuss this issue a little here. Assume that W is a given and fixed weight. Following Izumisawa and Kazamaki (1977), we say that W satisfies Muckenhoupt's condition A_p (where $1 < p < \infty$ is a fixed parameter), if

$$[W]_{A_p} := \sup_\tau \left\|\mathbb{E}\left[\left\{W_\tau/W_\infty\right\}^{1/(p-1)}\middle|\mathcal{F}_\tau\right]^{p-1}\right\|_{L^\infty} < \infty, \tag{1.2}$$

where the supremum is taken over the class of all adapted stopping times. It turns out that the weak-type estimate

$$\|X^*\|_{L^{p,\infty}(W)} \leq c_{p,[W]_{A_p}}\|X_\infty\|_{L^p(W)} \tag{1.3}$$

holds for all martingales X (with some constant $c_{p,[W]_{A_p}}$ depending only on the parameters indicated) if and only if W satisfies (1.2). Here we use the notations $\|X_\infty\|_{L^p(W)} = \left(\mathbb{E}|X_\infty|^p W_\infty\right)^{1/p}$ and $\|X^*\|_{L^{p,\infty}(W)} = \sup_{\lambda>0}\lambda[W(X^*\geq\lambda)]^{1/p}$ for the weighted strong and weak weighted pth norms of X (for $A \in \mathcal{F}$, we write $W(A) = \int_A W_\infty \mathrm{d}\mathbb{P}$). A similar phenomenon occurs in the context of strong type inequalities: the estimate

$$\|X^*\|_{L^p(W)} \leq C_{p,[W]_{A_p}}\|X_\infty\|_{L^p(W)}$$

holds for all X with some $C_{p,[W]_{A_p}}$ independent of X if and only if W is an A_p weight. These results, proved by Izumisawa and Kazamaki (1977), are in perfect correspondence with the classical theorems of Muckenhoupt concerning weighted inequalities for the Hardy–Littlewood maximal function on $\mathbb{R}^d$: cf. Muckenhoupt (1972).

We will provide the proof of the weighted counterpart of (1.1). In fact, we will establish a much stronger result: we will identify the best constant involved in this weighted estimate. To describe this constant, we need to introduce some auxiliary parameters. For the geometric interpretation of these objects, we refer the reader to Fig. 1. Let $c \geq 1$ and $1 < p < \infty$ be fixed. Then the line, tangent to the curve $x_1x_2^{p-1} = c$ at the point $(1, c^{1/(p-1)})$, intersects the curve $x_1x_2^{p-1} = 1$ at one point (if $c = 1$) or two points (if $c > 1$). Take the intersection point with larger x_1-coordinate, and denote this coordinate by $1 + d(p, c)$. Formally, $d = d(p, c)$ is the unique number in $[0, p-1)$ satisfying the equation

$$c(1+d)(p-1-d)^{p-1} = (p-1)^{p-1}. \tag{1.4}$$

We are ready to state the main result of the paper.