Statistics and Probability Letters 127 (2017) 33–41

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

On beta distributed limits of iterated linear random functions

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ARTICLE INFO

Article history:
Received 7 August 2016
Received in revised form 20 November 2016
Accepted 21 March 2017
Available online 31 March 2017

MSC:
primary 62E15
secondary 60J05
60H25
60D05

Keywords:
Iterated random functions
Nested interval schemes
Markov chains
Interval splitting
Random equations
Beta distribution

ABSTRACT

We consider several special cases of iterations of random i.i.d. linear functions with beta distributed fixed points. When iterated in a backward direction we obtain a nested interval scheme, whilst the forward direction generates an ergodic Markov chain. Our approach involves relating the random equation satisfied by the beta distributed fixed point to a random equation with a gamma distributed fixed point. The paper extends many results available in the existing literature.

1. Introduction

In this note we study iterations of linear random mappings

$$F_n := f_{A_n,B_n}, \quad n \geq 1, \tag{1}$$

where f_{A_n,B_n} are random elements of the family

$$\{f_{a,b}(x) = ax + b(1-x) : (a,b) \in [0,1]^2\},$$

$(A_1, B_1), (A_2, B_2), \ldots$ being an i.i.d. sequence of $[0,1]^2$-valued random vectors with a given distribution μ. The *forward iteration* of the mappings is given by the compositions

$$X_n(\cdot) := F_n \circ F_{n-1} \circ \cdots \circ F_1(\cdot), \quad n \geq 1,\ X_0(x) \equiv x, \tag{2}$$

while the *backward iteration* of the mappings is given by

$$Y_n(\cdot) := F_1 \circ F_2 \circ \cdots \circ F_n(\cdot), \quad n \geq 1,\ Y_0(x) \equiv x. \tag{3}$$

The general theory of forward and backward iterations of i.i.d. random functions is well-established (see e.g. Chamayou and Letac, 1991; Diaconis and Freedman, 1999; Letac, 1986). Other general results on iterated random functions are also found

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http://dx.doi.org/10.1016/j.spl.2017.03.021
0167-7152/

in for instance (Dyszewski, 2016; Rachev and Samorodnitsky, 1995; Stenflo, 2001) and references therein. Here we will only summarise the results relevant to this study.

It is clear that $X_n(\cdot)$ has the same distribution as $Y_n(\cdot)$ for each n. However, the properties of the forward and backward processes are very different. Under a broad contraction condition, for any fixed $x \in [0, 1]$, the forward sequence $\{X_n(x)\}_{n\geq 0}$ is an ergodic Markov chain, while the backward sequence $\{Y_n(x)\}_{n\geq 0}$ converges a.s. and is not in general a Markov chain (see e.g. Diaconis and Freedman, 1999). In addition, the limiting distributions for the two sequences coincide due to the following contraction principle (see e.g. Proposition 1 in Chamayou and Letac, 1991). If $Y := \lim_{n\to\infty} Y_n(x)$ a.s. exists and does not depend on x (in which case we will say that F_1 is a contraction), then the Markov chain $\{X_n(x)\}_{n\geq 0}$ is ergodic with stationary distribution given by the law of Y.

The forward process (2) is a special case of the random recurrence

$$V_n = D_n V_{n-1} + C_n, \quad n \geq 1, \tag{4}$$

where (D_n, C_n) are i.i.d. $\mathbb{R}^2$-valued random vectors. If V_n converges in distribution, the limiting distribution is the solution to the perpetuity equation

$$V \stackrel{d}{=} DV + C, \quad V \text{ independent of } (D, C) \stackrel{d}{=} (D_1, C_1). \tag{5}$$

A key reference on the perpetuity recurrence (4) is Vervaat (1979), where sufficient conditions for the existence and uniqueness of the limiting distribution of V_n as $n \to \infty$ are given. In particular, if the condition

$$\sum_{k=1}^{n} \log |D_k| \stackrel{d}{\to} -\infty \tag{6}$$

holds, then Lemma 1.5 in Vervaat (1979) implies that any solution V of (5) is unique in distribution and V_n converges in distribution to V for all V_0. In the special case of the forward process (2) generated by the i.i.d. mappings (1) it is clear that condition (6) is satisfied when

$$\mathbb{P}(|A_1 - B_1| = 1) < 1, \tag{7}$$

and this is precisely the condition required for F_1 to be a contraction.

In our case, starting from a fixed $x \in [0, 1]$ and writing $X_n = X_n(x)$, the forward iteration (2) is given by

$$X_n = (A_n - B_n)X_{n-1} + B_n, \quad n \geq 1. \tag{8}$$

If F_1 is a contraction, the unique stationary distribution P of the forward process (8) satisfies the random equation

$$X \stackrel{d}{=} AX + B(1 - X), \quad X \sim P \text{ independent of } (A, B) \stackrel{d}{=} (A_1, B_1), \tag{9}$$

where $X \sim P$ denotes that X has distribution P. Therefore, the stationary distribution of the forward process and the limiting distribution of the backward process coincide with the law of solution to (9).

We can interpret the forward process (2) in terms of the movement of a particle in $[0, 1]$, where X_n is the location of the particle at time n, and the particle moves from X_{n-1} to X_n by (8). In particular, we will consider those models where, for each n, at least one of the following holds: one has $A_n = 1$ (which would imply a move to the right) or $B_n = 0$ (implying a move to the left). Several special cases of models of this type (where, at each step, a particle at $X_n \in [0, 1]$ moves at time $n+1$ either left to a random point in $[0, X_n]$, or right to a random point in $[X_n, 1]$) have been shown to have beta distributed stationary distributions (see e.g. DeGroot and Rao, 1963; Diaconis and Freedman, 1999; Stoyanov and Pacheco-Gonzalez, 2008; Stoyanov and Pirinsky, 2000). See also McKinlay and Borovkov (2016), Pacheco-Gonzalez (2009) and Ramli and Leng (2010), where an extension of this model to the case when the direction of the next move is a function of the particles current location was considered.

The forward process has also been has been applied to improving the estimation of unobservable signals by adding Markovian noise induction (Iacus and Negri, 2003), while the extended version studied in McKinlay and Borovkov (2016) and Ramli and Leng (2010) was applied to a robot coverage algorithm which could equally apply to the models presented below. In addition, the extended model was used to model survival probabilities in a metapopulation model (McVinish et al., 2016).

We will interpret the backward process (3) as generating a sequence of nested intervals $\mathrm{I}_0 = [0, 1] \supset \mathrm{I}_1 \supset \mathrm{I}_2 \supset \cdots$ given by the ranges of the corresponding backward mappings $Y_0, Y_1, Y_2, \ldots$ (i.e. Y_n maps $[0, 1]$ onto I_n), and consider only the case when $|\mathrm{I}_n| \to 0$ as $n \to \infty$ (i.e. when F_1 is a contraction). A nested interval scheme of this type was studied in Johnson and Kotz (1995) and Kennedy (1988), where the *limiting location* Y was shown to be beta distributed.

In some of the cases we consider, the nested interval scheme generated by the backward iteration (3) is an interval splitting scheme of the following type. Choose a random (not necessarily uniformly distributed) *splitting point* S_1 in $[0, 1]$ and select according to a given (possibly random) rule one of the subintervals $[0, S_1]$, $[S_1, 1]$. Continuing this procedure in the same way independently on the chosen subinterval, we obtain a random sequence $\{S_n\}_{n\geq 1}$. When the law of the splitting point S_1 is not concentrated on $\{0, 1\}$, the length of the chosen interval tends to zero a.s. as $n \to 0$, and therefore S_n converges a.s. to a random variable Y. In several cases considered previously, this limiting random variable Y turns out to