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Computing an expected hitting time for the 3-urn Ehrenfest model via electric networks

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ABSTRACT

We study a three-urn version of the Ehrenfest model. This model can be viewed as a simple random walk on the graph represented by $\mathbb{Z}_3^M$, the M-fold direct product of the cyclic group $\mathbb{Z}_3$ of order 3, where M is the total number of balls distributed in the three urns. We build an electric network by placing a unit resistance on each edge of the graph. We then apply a series of circuit analysis techniques, including the series and parallel circuit laws and the Delta-Y transformation, to establish shorted triangular resistor networks. A recurrence relation is derived for the effective resistance between two corner vertices of the triangular resistor networks. The recurrence relation is then used to obtain an explicit formula for the expected hitting time between two extreme states where all balls reside in one of the three urns.

1. Introduction

We extend the two-urn Ehrenfest model by introducing a third urn. In this three-urn Ehrenfest model, M balls are distributed among urns 0, 1, and 2. At time $t = 1, 2, 3 \dots$, a ball is chosen at random, removed from the urn it resides in, and placed in one of the other two urns equally likely. As described in Bingham (1991), the process can be modeled as a random walk on a graph at two different levels.

1.1. Two model descriptions

In the "full description" we use M-tuples $\mathbf{Z}_t = (Z_{1,t}, Z_{2,t}, \dots, Z_{M,t})$ of ternary numbers to track the location of each ball at time t. That is, for $i = 1, 2, \dots, M$, $Z_{i,t}$ assumes a value of 0, 1, or 2 indicating that the ith ball is in urn 0, 1, or 2, respectively, at time t. Because each ball is chosen at random and placed in one of the other two urns with an equal probability, the process $\{\mathbf{Z}_t\}$ can be viewed as a simple random walk on the graph represented by $\mathbb{Z}_3^M$, the M-fold direct product of the cyclic group $\mathbb{Z}_3$ of order 3. The graph has 3^M vertices and $3^M M$ edges in total. Each vertex v is adjacent to $2M$ vertices whose M-tuple differs from v only by a single component.

On the other hand, in the "reduced description" we use vectors (X_t, Y_t) to keep track of the numbers of balls in urn 1 and urn 2, respectively, at time t. In this case, the vertices of the graph representing the three-urn Ehrenfest model are the

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integer lattice points (x, y) in the triangular grid bounded by three constraints: $0 \le x \le M$; $0 \le y \le M$; $0 \le x + y \le M$. The process $\{(X_t, Y_t)\}$ is a random walk on those lattice points, and in fact, a Markov chain with the set of the lattice points as its state space.

1.2. Hitting times and electric networks

In both model descriptions, we refer to the graph vertices as *states*. For a random walk on a graph, the *hitting time* (or *first passage time*) from state a to state b is the minimum number of steps the random walk takes to reach vertex b for the first time when the random walk initially starts at vertex a. The expected value of such a hitting time is denoted by $\mathbb{E}_a T_b$. The expected *commute time* between states a and b is $\mathbb{E}_a T_b + \mathbb{E}_b T_a$.

The expected hitting times associated with the two-urn Ehrenfest chain are well known. Blom (1989) expresses the expected hitting time between two adjacent states as a definite integral based on a recurrence relation. Bingham (1991) focuses on fluctuation theory and uses generating functions to compute the expected hitting time between the two extreme states. Palacios (1993, 1994) derives closed-form formulas for the expected hitting times via electric networks. One of the key ideas employed by Palacios through electric networks is based on a result due to Chandra et al. (1989), which is stated below.

Lemma 1 (*Chandra et al., 1989*)**.** *Let $G = (V, E)$ be a simple connected graph. Let $\mathcal{N}(G)$ be the electric network built by placing a node at each vertex in V and a one-ohm resistor on each edge in E. For each pair of vertices a and b in V, let $R_{a,b}$ denote the effective resistance between the two given nodes in $\mathcal{N}(G)$. Then for the simple random walk on the graph $G = (V, E)$, the expected commute time from vertex a to vertex b is $\mathbb{E}_a T_b + \mathbb{E}_b T_a = 2\,|E|\,R_{a,b}$, where $|E|$ is the cardinality of the edge set E.*

1.3. Our main result and approach

The expected hitting times associated with a multiple-urn Ehrenfest model do not appear to be in the literature. In this presentation, we will prove the following main result for the three-urn Ehrenfest model.

Main result. For the three-urn Ehrenfest model with M balls, $M \ge 1$, the expected hitting time from the state "urn 0 has all M balls" to the state "urn 1 has all M balls" is

$$\frac{2M}{3} \sum_{k=1}^{M} \frac{3^k}{k}.$$

The hitting time in our investigation involves three extreme states: All M balls are in urn 0, urn 1, or urn 2. We label the three extreme states by $\mathbf{a}_0$, $\mathbf{a}_1$, and $\mathbf{a}_2$, respectively. Our objective is to compute $\mathbb{E}_{\mathbf{a}_0} T_{\mathbf{a}_1}$ using the electric network approach. In addition to Lemma 1 and the series and parallel circuit laws used in the process of shorting an electric network, we will employ the *Delta-Y* transformation. The application of Delta-Y transformation to random walks on a graph is not new. Our colleagues in physics have drawn our attention to the Delta-Y transformation used by Wu et al. (2011) on studying a simple random walk on dual Sierpinski gaskets.

2. Computing $\mathbb{E}_{\mathbf{a}_0} T_{\mathbf{a}_1}$

Since the three-urn Ehrenfest model can be regarded as a simple random walk on the graph represented by $\mathbb{Z}_3^M$, we can build an electric network by placing a node at each vertex and a one-ohm resistor on each edge and apply Lemma 1 to compute the expected commute time between $\mathbf{a}_0$ and $\mathbf{a}_1$ as

$$\mathbb{E}_{\mathbf{a}_0} T_{\mathbf{a}_1} + \mathbb{E}_{\mathbf{a}_1} T_{\mathbf{a}_0} = 2\,|E|\,R_{\mathbf{a}_0,\mathbf{a}_1}.$$

Due to symmetry, we have $\mathbb{E}_{\mathbf{a}_0} T_{\mathbf{a}_1} = \mathbb{E}_{\mathbf{a}_1} T_{\mathbf{a}_0}$. With the edge number $|E|$ equal to $3^M M$, the expected hitting time from $\mathbf{a}_0$ to $\mathbf{a}_1$ is

$$\mathbb{E}_{\mathbf{a}_0} T_{\mathbf{a}_1} = 3^M M \times R_{\mathbf{a}_0,\mathbf{a}_1}. \tag{1}$$

So it remains to find the effective resistance $R_{\mathbf{a}_0,\mathbf{a}_1}$ between $\mathbf{a}_0$ and $\mathbf{a}_1$ for computing the expected hitting time $\mathbb{E}_{\mathbf{a}_0} T_{\mathbf{a}_1}$.

2.1. Shorting the full-description electric network

For the electric network built on the simple random walk on $\mathbb{Z}_3^M$, we use an idea by Palacios (1994) to apply a unit voltage between $\mathbf{a}_0$ and $\mathbf{a}_1$ such that the voltage at $\mathbf{a}_0$ is 1 and the voltage at $\mathbf{a}_1$ is 0. Then all vertices represented by the M-tuples with the same number of 1's and the same number of 2's will have the same voltage and can be shorted. There are $\binom{M}{x}\binom{M-x}{y}$ such M-tuples with x 1's and y 2's, and those vertices are shorted into a vertex represented by the integer lattice point (x, y) with $0 \le x,\ y,\ x + y \le M$. This shorting process yields a triangular grid, which is the reduced description of the three-urn Ehrenfest model. We now find the edge resistances between vertex (x, y) and its nearest neighbors in this